4.4

Determine type & stability of C.P.
gen. soln.
sketch

pg. 151 (3) $y_1' = y_2$
$y_2' = -9y_1$

$\vec{y}' = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \vec{y} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -9 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 9 = 0 \Rightarrow \lambda^2 = -9$

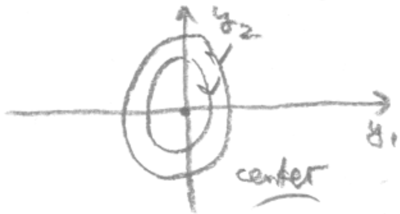
$\Rightarrow \underline{\lambda = \pm 3i}$

$p = \lambda_1 + \lambda_2 = +3i + (-3i) = 0$
$q = \lambda_1 \lambda_2 = +3i(-3i) = -9i^2 = -9(-1) = 9$ $\Rightarrow$ critical point is a center, stable $\quad q > 0$, $p = 0$

$\lambda_1 = 3i \quad -3ix_1 = -x_2 \Rightarrow x_2 = 3ix_1 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 3i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$\Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 3 \end{bmatrix} \qquad e^{i\mu t} = \cos \mu t + i \sin \mu t$

$\Rightarrow \vec{y} = c_1 \begin{bmatrix} \cos 3t \\ -3\sin 3t \end{bmatrix} + c_2 \begin{bmatrix} \sin 3t \\ 3\cos 3t \end{bmatrix} \Rightarrow \begin{array}{l} y_1 = c_1 \cos 3t + c_2 \sin 3t \\ y_2 = -3c_1 \sin 3t + 3c_2 \cos 3t \end{array}$



(5). $y_1' = -2y_1 + 2y_2$
$y_2' = -2y_1 - 2y_2$ $\Rightarrow \vec{y}' = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \vec{y}$

$\Rightarrow \begin{vmatrix} -2-\lambda & 2 \\ -2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (-2-\lambda)^2 + 4 = 0 \Rightarrow 4 - 2(-2\lambda) + \lambda^2 + 4 = 0$
$\Rightarrow \lambda^2 + 4\lambda + 8 = 0$

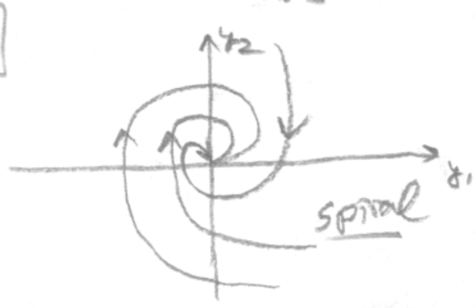
$\Rightarrow \lambda = \frac{-4 \pm \sqrt{16-32}}{2} = -2 \pm \frac{1}{2}\sqrt{16}\sqrt{-1} = \boxed{-2 \pm i2}$

$p = (-2+i2) + (-2-i2) = -4$
$q = (-2+i2)\cdot(-2-i2) = 4 - 4i^2 = 8$ $\quad$ spiral point, stable

$\lambda_1 = -2 + i2 \quad -2 - (-2+2i)x_1 = -x_2 \Rightarrow x_2 = 2ix_1 \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 2i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\Rightarrow \vec{y} = c_1 e^{-2t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}$

$y_1 = e^{-2t}(c_1 \cos 2t + c_2 \sin 2t)$
$y_2 = e^{-2t}(c_2 \cos 2t - c_1 \sin 2t)$



4.4

pg 151 (4) $y_1' = y_1 + 2y_2$
$y_2' = 2y_1 + y_2$
$\Rightarrow \vec{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{y} \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2 - 4 = 0 \Rightarrow 1 - 2\lambda + \lambda^2 - 4 = 0 \Rightarrow \lambda^2 - 2\lambda - 3 = 0 \Rightarrow (\lambda-3)(\lambda+1) = 0$

$\Rightarrow \lambda_1 = 3; \lambda_2 = -1 \Rightarrow p = \lambda_1 + \lambda_2 = 2$, $q = \lambda_1 \lambda_2 = -3$ saddle point, unstable

$\lambda_1 = 3$: $\begin{bmatrix} 1-3 & 2 \\ 2 & -2 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow 2x_1 = 2x_2 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$: $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow x_1 = -x_2 \Rightarrow \vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\Rightarrow \vec{y} = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow y_1 = c_1 e^{3t} - c_2 e^{-t}$
$y_2 = c_1 e^{3t} + c_2 e^{-t}$

saddle

(11) $y'' + 2y' + 2y = 0$
$\Rightarrow y'' = -2y' - 2y$

$y_1 = y$, $y_2 = y' = y_1'$, $y_3 = y'' = y_2'$ $\Rightarrow$ 1) $y_1' = y_2$
2) $y_2' = -2y_2 - 2y_1$

$\Rightarrow \vec{y}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \vec{y} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(-2-\lambda) + 2 = 0$

$\Rightarrow 2\lambda + \lambda^2 + 2 = 0$ or $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \frac{1}{2}\sqrt{4}\sqrt{-1}$

$\Rightarrow \lambda = -1 \pm i \Rightarrow p = (-1+i) + (-1-i) = -2$
$q = (-1+i)(-1-i) = 1 - i^2 = 2$ stable, spiral (attractive)

$\mu = 1$
$r = -1$

$e^{i\mu t} = \cos\mu t + i\sin\mu t$; $e^{i\mu t} e^{rt} (\vec{a} + i\vec{b}) = e^{-t}(\vec{a}\cos t - \vec{b}\sin t) +$
$+ e^{-t} i (\vec{b}\cos t + \vec{a}\sin t)$

or $y = e^{-t}(\vec{a}\cos t + \vec{b}\sin t)$

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4.5

pg. 159

Find loc. type of c.p.

(4). $y_1' = 4y_1 - y_1^2 = y_1(4-y_1) = f_1$

$y_2' = y_2 = f_2$

$f_2 = y_2 = 0 \Rightarrow y_2 = 0$

$f_1 = 4y_1 - y_1^2 \Rightarrow y_1(4-y_1) = 0 \Rightarrow y_1 = 0$ or $y_1 = 4$

• 2 c.p.: $\overset{y_1,\,y_2}{(0,0)}$, $\overset{y_1,\,y_2}{(4,0)}$

for (0,0): $\vec{f}(\vec{y}) = \vec{f}(\vec{y_0}) + \nabla\vec{f}(\vec{y_0})(\vec{y}-\vec{y_0}) + \vec{R}(\vec{y})$

$$\nabla\vec{f}(\vec{y_0}) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1}(\vec{y_0}) & \frac{\partial f_1}{\partial y_2}(\vec{y_0}) \\ \frac{\partial f_2}{\partial y_1}(\vec{y_0}) & \frac{\partial f_2}{\partial y_2}(\vec{y_0}) \end{bmatrix}$$

$$\Rightarrow \nabla\vec{f} = \begin{bmatrix} 4-2y_1 & 0 \\ 0 & 1 \end{bmatrix}$$

• $\nabla\vec{f}(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{vmatrix} 4-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(4-\lambda) = 0$

$\Rightarrow \lambda_1 = 1;\ \lambda_2 = 4 \Rightarrow p = \lambda_1 + \lambda_2 = 5 > 0 \Rightarrow$ unstable node

$q = \lambda_1\lambda_2 = 4$

• $\nabla\vec{f}(4,0) = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{vmatrix} -4-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-4-\lambda) = 0$

$y_1 = 4$
$y_2 = 0$

$\Rightarrow \lambda_1 = 1$; $-4-\lambda = 0 \Rightarrow -4 = \lambda$

$\lambda_2 = -4$

$\Rightarrow p = -3$ unstable; saddle

$q = -4 < 0$

4.5

pg. 159

(4). $y_1' = -y_1 + y_2 - y_2^2 = f_1$ Find loc. / type of c.p.

$y_2' = -y_1 - y_2 = f_2$

$f_1 = 0 = -y_1 + y_2 - y_2^2 = -y_1 + y_2(1-y_2)$ $f_2 = 0 = -y_1 - y_2 \Rightarrow -y_1 = y_2$ or $\underline{y_1 = -y_2}$ plug in f_1

$\Rightarrow y_2 + y_2 - y_2^2 = 0$

$\Rightarrow 2y_2 - y_2^2 = 0 \Rightarrow y_2(2-y_2) = 0$

$\Rightarrow y_2 = 0 \Rightarrow y_1 = 0$ c.p.: $(0,0)$

$\Rightarrow y_2 = 2 \Rightarrow y_1 = -2$ c.p.: $(-2,2)$

$$\nabla \vec{f}(\vec{y_0}) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1}(\vec{y_0}) & \frac{\partial f_1}{\partial y_2}(\vec{y_0}) \\ \frac{\partial f_2}{\partial y_1}(\vec{y_0}) & \frac{\partial f_2}{\partial y_2}(\vec{y_0}) \end{bmatrix} = \begin{bmatrix} -1 & 1-2y_2 \\ -1 & -1 \end{bmatrix}$$

- $\nabla \vec{f}(0,0) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$; $\begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (-1-\lambda)^2 + 1 = 0$

  $y_1 = y_2 = 0$

$\Rightarrow 1 - 2(-1)(\lambda) + \lambda^2 + 1 = 0 \Rightarrow \lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \frac{1}{2}\sqrt{4}\sqrt{-1}$

$\Rightarrow \underline{\lambda_{1,2} = -1 \pm i}$ spiral (attractive) $p = \lambda_1 + \lambda_2 = (-1+i) + (-1-i) = -2 < 0$ stable

$q = \lambda_1 \lambda_2 = (-1+i)(-1-i) = 1 - i^2 = 2 > 0$

- $\vec{\nabla} f(-2,2) = \begin{bmatrix} -1 & -3 \\ -1 & -1 \end{bmatrix}$; $\begin{vmatrix} -1-\lambda & -3 \\ -1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (-1-\lambda)^2 - 3 = 0$

  $y_1 = -2$
  $y_2 = 2$

$\Rightarrow \lambda^2 + 2\lambda + 1 - 3 = 0 \Rightarrow \lambda^2 + 2\lambda - 2 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4+8}}{2}$

$\Rightarrow \lambda = -1 \pm \frac{1}{2}\sqrt{12} \Rightarrow \lambda_1 = -1 + .5\sqrt{12}$, $\lambda_2 = -1 - .5\sqrt{12}$

$p = -1 + \cancel{.5\sqrt{12}} - 1 - \cancel{.5\sqrt{12}} = -2 < 0$

$q = (-1 + .5\sqrt{12})(-1 - .5\sqrt{12}) = 1 - .5^2 \cdot 12 = -2 < 0$ unstable saddle point

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4.5

pg. 159

convert to ODE
find loc. & type of c.p.

$y'' + \cos y = 0$

(11).

$\Rightarrow y'' = -\cos y$

$\Rightarrow y_2' = -\cos y_1 = f_2$

$y_1' = y_2 = f_1$

$y_1 = y,$
$y_2 = y' = y_1'$
$y_3 = y'' = y_2'$

$\Rightarrow f_1 = 0 = y_2 \Rightarrow y_2 = 0$
$f_2 = 0 = -\cos y_1 \Rightarrow y_1 = \pm \frac{\pi}{2}$

$\Rightarrow \boxed{\text{c.p.s: } \left(\pm\frac{\pi}{2} \pm 2n\pi, 0\right)}$

$$\nabla \vec{f}(\vec{y_0}) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1}(\vec{y_0}) & \frac{\partial f_1}{\partial y_2}(\vec{y_0}) \\ \frac{\partial f_2}{\partial y_1}(\vec{y_0}) & \frac{\partial f_2}{\partial y_2}(\vec{y_0}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sin y_1 & 0 \end{bmatrix}$$

$$\nabla \vec{f}\left(\frac{\pi}{2}, 0\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda_1 = 1, \ \lambda_2 = -1$$

$\Rightarrow p = \lambda_1 + \lambda_2 = 0$
$q = \lambda_1 \lambda_2 = -1 < 0 \Rightarrow$ $\boxed{\text{unstable saddle point}}$

$$\nabla \vec{f}\left(-\frac{\pi}{2}, 0\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ; \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$$

$\Rightarrow p = 0$
$q = i \cdot (-i) = -i^2 = 1$ center stable

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4.6

pg. 163

(3)
$y_1' = y_2 + e^{3t}$
$y_2' = y_1 - 3e^{3t}$ $\Rightarrow \vec{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{y} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{3t}$ $\quad \vec{y}' = A\vec{y} + \vec{g}$

$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow (\lambda - 1)(\lambda + 1) = 0 \Rightarrow \underline{\lambda_1 = 1, \lambda_2 = -1}$

$\underline{\lambda_1 = 1}$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow x_1 = x_2 \Rightarrow \underline{\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$\underline{\lambda_2 = -1}$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow x_1 = -x_2 \Rightarrow \underline{\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

$\Rightarrow \vec{y}_h = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\vec{y}_p = e^{3t} \vec{v} \Rightarrow \vec{y}_p' = 3e^{3t} \vec{v} = e^{3t} A\vec{v} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{3t} \Rightarrow 3\vec{v} - A\vec{v} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} = 0$
& $\vec{y}' = A\vec{y} + \vec{g}$

$\Rightarrow (3I - A)\vec{v} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} = 0 \Rightarrow \left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\Rightarrow \left[\begin{array}{cc|c} 3 & -1 & 1 \\ -1 & 3 & -3 \end{array}\right] \uparrow \times 3 \Rightarrow \left[\begin{array}{cc|c} 0 & 8 & -8 \\ -1 & 3 & -3 \end{array}\right]$

i) $0v_1 + 8v_2 = -8 \Rightarrow 0v_1 + v_2 = -1$
$\Rightarrow v_2 = -1 - 0v_1 \Rightarrow \underline{v_2 = -1}$

ii) $-1v_1 + 3v_2 = -3$
$\Rightarrow 3v_2 + 3 = v_1 \Rightarrow v_1 = 0$

$\Rightarrow \vec{v} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \vec{y}_p = e^{3t} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\Rightarrow \vec{y} = \vec{y}_h + \vec{y}_p = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

---

or $y_1 = c_1 e^{t} - c_2 e^{-t}$
$y_2 = c_1 e^{t} + c_2 e^{-t} - e^{3t}$

4.6

Pg. 163 $\quad y_1' = y_2 + 6e^{2t} \qquad y_1(0) = 1,\ y_2(0) = 0$

(11). $\quad y_2' = y_1 - e^{2t}$

$\Rightarrow \vec{y}' = A\vec{y} + g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{y} + \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{2t}$

$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = -1$

$\lambda_1 = 1$: $\quad -x_1 = -x_2 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_2 = -1$: $\quad x_1 = -x_2 \Rightarrow \vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\Rightarrow \vec{y}_h = c_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\vec{y}_p = e^{2t}\vec{u} \Rightarrow \vec{y}_p' = 2e^{2t}\vec{u} = e^{2t}A\vec{u} + \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{2t}$

$\Rightarrow 2\vec{u} - A\vec{u} = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \Rightarrow (2I - A)\vec{u} = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 6 \\ -1 & 2 & -1 \end{array}\right] \uparrow \times 2$

$\Rightarrow \left[\begin{array}{cc|c} 0 & 3 & 4 \\ -1 & 2 & -1 \end{array}\right] \Rightarrow$ i) $3u_2 = 4 \Rightarrow u_2 = 4/3$

ii) $-u_1 + 2u_2 = -1 \Rightarrow -u_1 + \frac{8}{3} = -1 \Rightarrow -3u_1 = -11 \Rightarrow u_1 = \frac{11}{3}$

$\Rightarrow \vec{u} = \begin{bmatrix} 11/3 \\ 4/3 \end{bmatrix} \Rightarrow \vec{y}_p = e^{2t}\begin{bmatrix} 11/3 \\ 4/3 \end{bmatrix}$

$\Rightarrow \vec{y} = \vec{y}_h + \vec{y}_p = c_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^{2t}\begin{bmatrix} 11/3 \\ 4/3 \end{bmatrix}$

$\Rightarrow y_1 = 1 = c_1 - c_2 + \frac{11}{3} \Rightarrow c_1 - c_2 = -\frac{8}{3} \Rightarrow c_1 = -\frac{8}{3} + c_2$

$y_2 = 0 = c_1 + c_2 + \frac{4}{3} \Rightarrow -\frac{8}{3} + 2c_2 + \frac{4}{3} = 0 \Rightarrow 2c_2 = \frac{4}{3} \Rightarrow 6c_2 = 4 \Rightarrow c_2 = \frac{4}{6} = \frac{2}{3}$

$\Rightarrow c_1 = -\frac{6}{3} = -2$

$\Rightarrow \vec{y} = -2e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{2}{3}e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^{2t}\begin{bmatrix} 11/3 \\ 4/3 \end{bmatrix}$

$y_1 = -2e^{t} - \frac{2}{3}e^{-t} + \frac{11}{3}e^{2t}$; $y_2 = -2e^{t} + \frac{2}{3}e^{-t} + \frac{4}{3}e^{2t}$