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Problem #3: Determine the type and stability of the critical point. Then find a real general solution and sketch or graph some of the trajectories in the phase plane. Show details of work.

$y_1' = y_2$ → $y_1' = 0y_1 + 1y_2$ → $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$y_2' = -9y_1$ → $y_2' = -9y_1 + 0y_2$

$\vec{A} - \lambda\vec{I} = \lambda^2 + 9 = 0$ → $(\lambda + 3i)(\lambda - 3i)$

$\therefore \lambda_1 = -3i \quad \lambda_2 = 3i$

$p = \lambda_1 + \lambda_2 = 0$

$q = \lambda_1 \lambda_2 = 9$ → From Table 4.1 and Table 4.2 → $\boxed{\text{Center and Stable}}$

$\Delta = (\lambda_1 - \lambda_2)^2 = -36$

Find equation for y_1 and y_2

General Solution Complex roots : $y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$ where $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$

for $\lambda = -3i$ → $\begin{bmatrix} 3i & 1 \\ -9 & 3i \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix}$ → $3i = -x$ → $\begin{bmatrix} 1 \\ -3i \end{bmatrix}$

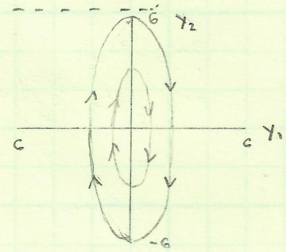
Note $a = 0$, $\omega = 3$

for $\lambda = 3i$ → $\begin{bmatrix} -3i & 1 \\ -9 & -3i \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix}$ → $3i = x$ → $\begin{bmatrix} 1 \\ 3i \end{bmatrix}$

$\therefore \vec{y}(t) = e^{-(a/2)t}(\cos\omega t + i\sin\omega t)$ $\quad \vec{y}(t) = e^{-(a/2)t}(\cos\omega t - i\sin\omega t)$

$$\boxed{\begin{aligned} y_1 &= B\sin(3t) + A\cos(3t) \\ y_2 &= -3A\sin(3t) + 3B\cos(3t) \end{aligned}}$$

See figure on the right



Problem #5: See Problem #3 directions

$y_1' = -2y_1 + 2y_2$ → $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$y_2' = -2y_1 - 2y_2$

$\vec{A} - \lambda\vec{I}$ → $\lambda^2 + 4\lambda + 8$ → $\dfrac{-4 \pm \sqrt{16-32}}{2}$ → $\lambda = -2 \pm 2i$ → $\lambda_1 = -2 + 2i$, $\lambda_2 = -2 - 2i$

$p = \lambda_1 + \lambda_2 = -4 \qquad q = \lambda_1\lambda_2 = 8 \qquad \Delta = (\lambda_1 - \lambda_2)^2 = -16$

→ from Table 4.1 and Table 4.2 → $\boxed{\text{Spiral Point , stable}}$

Find general solution Complex roots : $y = e^{-ax/2}(A\cos\omega t + B\sin\omega t)\,x$ $\qquad a = 4 \quad \omega = 2$

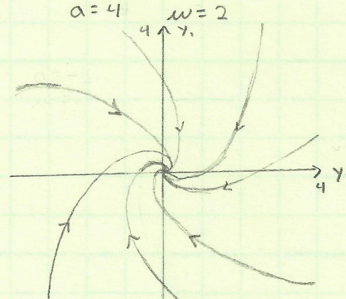
for $\lambda = -2+2i$ → $\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix}$ → $x = i$ → $\begin{bmatrix} 1 \\ i \end{bmatrix}$

for $\lambda = -2-2i$ → $\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix}$ → $x = -i$ → $\begin{bmatrix} 1 \\ -i \end{bmatrix}$

for $\lambda = -2+2i$ → $\boxed{y(t) = e^{-2t}(A\cos 2t + B\sin 2t)}$

for $\lambda = -2-2i$ → $\boxed{y(t) = e^{-2t}(B\cos 2t - A\sin 2t)}$

See figure on right



Problem #7 : See Problem #3 directions

$y_1' = y_1 + 2y_2$ → $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$y_2' = 2y_1 + y_2$

$\vec{A} - \lambda\vec{I}$ → $\lambda^2 - 2\lambda + -3$ → $\dfrac{2 \pm \sqrt{4+12}}{2} = 1 \pm 2$ → $\lambda_1 = -1 \quad \lambda_2 = 3$

$p = \lambda_1 + \lambda_2 = 2 \qquad q = \lambda_1\lambda_2 = -3 \qquad \Delta = (\lambda_1 - \lambda_2)^2 = 16$

→ from Table 4.1 and Table 4.2 ↔ $\boxed{\text{Saddle Point , Unstable}}$

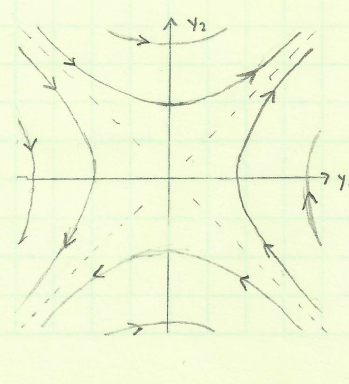
General solution : $c_1\vec{x}_1 e^{\lambda t} + c_2\vec{x}_2 e^{\lambda t}$

for $\lambda = -1$ → $x_1 = [\,1\ -1\,]^T$

for $\lambda = 3$ → $x_2 = [\,1\ \ 1\,]^T$

$$\boxed{y(t) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{3t}}$$

See figure on right



Problem #11: Solve $y'' + 2y' + 2y = 0$, What kind of curves are the trajectories

Let $y_1 = y$ and $y_2 = y'$ $\rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\rightarrow y_1' = y_2$ and $y_2' = -2y_1 + -2y_2$

$\vec{A} - \lambda\vec{I} = \lambda^2 + 2\lambda + 2 \rightarrow \frac{-2 \pm \sqrt{4-8}}{2} \rightarrow 1 \pm i \rightarrow \lambda_1 = -1 + i \quad \lambda_2 = -1 - i$

$P = \lambda_1 + \lambda_2 = -2 \quad Q = \lambda_1\lambda_2 = 2 \quad \Delta = (\lambda_1 - \lambda_2)^2 = -4$

from Table 4.1 and Table 4.2 $\rightarrow$ $\boxed{\text{Stable and Attractive Spiral Point}}$

General solution $y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$

$\therefore \boxed{y(t) = e^{-t}(A \cos t + B \sin t)}$

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Problem #4: Find the location and type of all critical points by linearization. Show the details of your work.

$y_1' = 4y_1 - y_1^2 \rightarrow y_1' = y_1(4 - y_1)$

$y_2' = y_2 \qquad y_2' = y_2$

Find critical Point ($y_1' = 0$ and $y_2' = 0$)

$0 = y_1(4 - y_1) \rightarrow y_1 = 0, 4$

$0 = y_2 \rightarrow y_2 = 0, 0$ $\rightarrow \therefore \boxed{\text{Critical points: } (0,0); (4,0)}$

At (0,0)

$y_1' = 4y_1$
$y_2' = y_2$ $\rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

$|\vec{A} - \lambda\vec{I}| = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) \rightarrow \lambda = 1, 4$

$P = \lambda_1 + \lambda_2 = 5 \quad q = \lambda_1\lambda_2 = 4 \quad \Delta = (\lambda_1 - \lambda_2)^2 = 9$

from Table 4.1 and 4.2 $\rightarrow$ $\boxed{\text{At } (0,0) \text{ Unstable Node}}$

At (4,0)

$y_1' = 4 + \tilde{y}_1 \rightarrow \tilde{y}_1' = (4 + \tilde{y}_1') - \tilde{y}_1' \rightarrow \tilde{y}_1' = -4\tilde{y}_1 \rightarrow \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$

$y_2' = \tilde{y}_2 \qquad \tilde{y}_2' = \tilde{y}_2' \qquad \tilde{y}_2' = \tilde{y}_2$

$|\vec{A} - \lambda\vec{I}| = \lambda^2 + 3\lambda - 4 = (\lambda + 4)(\lambda - 1) \rightarrow \lambda = 1, -4$

$P = \lambda_1 + \lambda_2 = -3 \quad q = \lambda_1\lambda_2 = -4 \quad \Delta = (\lambda_1 - \lambda_2)^2 = 25$

from Table 4.1 and 4.2 $\rightarrow$ $\boxed{\text{At } (4,0) \text{ Unstable Saddle Point}}$.

Problem #7: Find the location and type of all critical points by linearization. Show details of your work.

$y_1' = -y_1 + y_2 - y_2^2 \rightarrow y_1' = -y_1 + y_2(1 - y_2)$

$y_2' = -y_1 - y_2 \qquad y_2' = -y_1 - y_2$

Find critical Point ($y_1' = 0$ $y_2' = 0$)

$0 = -y_1 + y_2(1 - y_2)$ $\rightarrow y_2 + y_2(1 - y_2) \rightarrow -y_2^2 + 2y_2 \rightarrow y_2^2 = 2y^2$

$0 = -y_1 - y_2 \rightarrow y_1 = -y_2 \qquad \rightarrow y_2 = 2, \ y_1 = -2$

$\boxed{\therefore \text{Critical Points } (0,0); (-2,2)}$

At (0,0)

$y_1' = -y_1 + y_2$
$y_2' = -y_1 - y_2$ $\rightarrow \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$

$|A - \lambda\vec{I}| = \lambda^2 + 2\lambda + 2 \rightarrow \frac{-2 \pm \sqrt{4-8}}{2} = 1 \pm i \rightarrow \lambda_1 = -1 - i, \ \lambda_2 = -1 + i$

$P = \lambda_1 + \lambda_2 = -2 \quad q = \lambda_1\lambda_2 = 2 \quad \Delta = (\lambda_1 - \lambda_2)^2 = -4$

from Table 4.1 and 4.2 $\rightarrow$ $\boxed{\text{At } (0,0) \text{ Stable and Attractive Spiral Point}}$

At (-2,2)

$y_1 = -2 + \tilde{y}_1 \rightarrow \tilde{y}_1' = -(-2 + \tilde{y}_1) + (2 + \tilde{y}_2)(1 - (2 - \tilde{y}_2)) \rightarrow \tilde{y}_1' = 2 - \tilde{y}_1 + (\cancel{-\tilde{y}_2^2} - 3\tilde{y}_2 - 2) \rightarrow \tilde{y}_1' = -\tilde{y}_1 - 3\tilde{y}_2$

$y_2 = 2 + \tilde{y}_2 \qquad \tilde{y}_2' = -(-2 + \tilde{y}_1) - (2 + \tilde{y}_2') \qquad \tilde{y}_2' = -\tilde{y}_1 - \tilde{y}_2 \qquad \tilde{y}_2' = -\tilde{y}_1 - \tilde{y}_2$

$\begin{bmatrix} -1 & -3 \\ -1 & -1 \end{bmatrix} \rightarrow |\vec{A} - \lambda\vec{I}| = \lambda^2 + 2\lambda - 2 \rightarrow \frac{-2 \pm \sqrt{4+8}}{2} \rightarrow 1 \pm \sqrt{3} \rightarrow \lambda_1 = -1 - \sqrt{3}, \ \lambda_2 = -1 + \sqrt{3}$

$P = \lambda_1 + \lambda_2 = -2 \quad q = \lambda_1\lambda_2 = -2 \quad \Delta = (\lambda_1 - \lambda_2)^2 = 9$

from Table 4.1 and 4.2 $\rightarrow$ $\boxed{\text{At } (-2,2) \text{ Stable and Attractive Saddle Point.}}$

Problem #11: Find the location and type of all critical points by first converting the ODE to a system and then linearizing it.

$y'' + \cos(y) = 0$ Let $\theta = y_1$ and $\theta' = y_2$

$\rightarrow y_1' = y_2 = 0$ $\rightarrow$ Infinitely many critical points. $\rightarrow \boxed{\left(\frac{\pi}{2} \pm n\pi, 0\right) = \text{critical point}}$ $n = 0, \pm 1, \pm 2 \ldots$

$y_2' = -\cos(y_1) = 0$

Considering $\left(\frac{\pi}{2}, 0\right)$ $\rightarrow$ set $y_1 = \theta - \frac{\pi}{2}$

Substitute $y_1' = y_2$

$y_2' = -\cos\left(\theta - \frac{\pi}{2}\right) = -\sin(\theta)$ By trigonometric functions

Maclaurin series: $-\sin(\theta) = -\sin(y_1) = -y_1 + \frac{1}{6}y_1^3 - \ldots \approx -y_1$

$\rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{matrix} y_1' = y_2 \\ y_2' = y_1 \end{matrix} \rightarrow \lambda^2 - 1 = (\lambda+1)(\lambda-1) \rightarrow \lambda_1 = -1 \quad \lambda_2 = 1$

$P = 0 \quad Q = -1 \quad \Delta = 4$

from table 4.1 and 4.2 $\rightarrow$ $\boxed{\text{At } \left(\frac{\pi}{2} \pm 2\pi n, 0\right) \text{ it is a saddle point}}$

Considering $\left(-\frac{\pi}{2}, 0\right)$ $\rightarrow$ set $y_1 = \theta + \frac{\pi}{2}$

Substitute $y_1' = y_2$

$y_2' = -\cos\left(\theta + \frac{\pi}{2}\right) = \sin(\theta)$ By trigonometric functions

Maclaurin Series $\sin(\theta) = \sin(y_1) = y_1 - \frac{1}{6}y_1^3 \ldots \approx y_1$

$\rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rightarrow \begin{matrix} y_1' = y_2 \\ y_2' = -y_1 \end{matrix} \rightarrow \lambda^2 + 1 = (\lambda + i)(\lambda - i) \rightarrow \lambda_1 = -i \quad \lambda_2 = i$

$P = 0 \quad q = 1 \quad \Delta = -4$

from table 4.1 and 4.2 $\rightarrow$ $\boxed{\text{At } \left(-\frac{\pi}{2} \pm 2\pi n, 0\right) \text{ it is a Center}}$

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Problem #3: Find a general solution. Show your work.

$y_1' = y_2 + e^{3t}$
$y_2' = y_1 - 3e^{3t}$ $\rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{3t}$

Solve for Homogenous Solution

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ Let $xe^{\lambda t}$ be the solution then $y' = \lambda x e^{\lambda t}$.

Recall $\vec{A}\vec{x} = \lambda\vec{x} \rightarrow y' = \vec{A}xe^{\lambda t}$

$|\vec{A} - \lambda\vec{I}| = \lambda^2 - 1 = (\lambda+1)(\lambda-1) \rightarrow \lambda_1 = 1, \lambda_2 = -1$

for $\lambda = 1$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

for $\lambda = -1$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

General solution for homogeneous Portion

$y' = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} \quad (1) \quad = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Solve for Particular solution

$y^{(p)} = \vec{Y}(t)\vec{u}(t)$ $\quad \vec{Y}(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}$

Substitute into solution $\vec{y}' = \vec{A}\vec{y} + \vec{g} \rightarrow \vec{Y}'\vec{u} + \vec{Y}\vec{u}' = \vec{A}\vec{Y}\vec{u} + \vec{g}$

Since $\vec{y}^{(1)}$ and $\vec{y}^{(2)}$ are solutions of the homogeneous solution we have

$\vec{y}^{(1)\prime} = \vec{A}\vec{y}^{(1)}, \quad \vec{y}^{(2)\prime} = \vec{A}\vec{y}^{(2)}$ thus $\vec{Y}' = \vec{A}\vec{Y}$

Combineing with 2 lines above reveals that $\vec{Y}\vec{u}' = \vec{g} \rightarrow \vec{u}' = \vec{Y}^{-1}\vec{g}$

$\vec{Y}^{-1} = \frac{1}{\det(A)}\begin{vmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{vmatrix} = \frac{1}{-2e^0}\begin{vmatrix} -e^{-t} & -e^{-t} \\ -e^t & e^t \end{vmatrix} = -\frac{1}{2}\begin{bmatrix} -e^{-t} & -e^{-t} \\ -e^t & e^t \end{bmatrix}$

$\rightarrow \vec{u}' = -\frac{1}{2}\begin{bmatrix} -e^{-t} & -e^{-t} \\ -e^t & e^t \end{bmatrix}\begin{bmatrix} e^{3t} \\ -3e^{3t} \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} -e^{2t} + 3e^{2t} \\ -e^{4t} - 3e^{4t} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 2e^{2t} \\ -4e^{4t} \end{bmatrix} = \begin{bmatrix} -e^{2t} \\ 2e^{4t} \end{bmatrix}$

$$\vec{u} = \int_0^t \begin{bmatrix} -e^{2t} \\ 2e^{4t} \end{bmatrix} dt = \begin{matrix} \frac{1}{2}(-e^{2t})\big|_0^t \\ \frac{2}{4}(e^{4t})\big|_0^t \end{matrix} \to \begin{matrix} -\frac{1}{2}e^{2t} + \frac{1}{2} \\ \frac{1}{2}e^{4t} - \frac{1}{2} \end{matrix} \to \frac{1}{2}\begin{bmatrix} -e^{2t} \\ e^{4t} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$y^{(P)} = \vec{Y}(t)\vec{u}(t) = \frac{1}{2}\begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}\begin{bmatrix} -e^{2t} \\ e^{4t} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -e^{3t} + e^{3t} \\ -e^{3t} - e^{3t} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} e^{t} - e^{-t} \\ e^{t} \quad e^{-t} \end{bmatrix} \to \frac{1}{2}\begin{bmatrix} 0 \\ -2e^{3t} \end{bmatrix} + \frac{1}{2}e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\to -e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{2}e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + -\frac{1}{2}e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad (2)$$

Combine Particular Solution and General solution.

Notice that the last two parts of the particular solution resemble the general solution. These get absorbed.

$$\boxed{\therefore\ y(t) = c_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + -e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

Problem #11 : Solve Showing Details

$$y_1' = y_2 + 6e^{2t} \qquad y_1(0) = 1 \qquad \to \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{2t}$$

$$y_2' = y_1 - e^{2t} \qquad y_2(0) = 0$$

Solve Homogeneous

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to |\vec{A} - \lambda\vec{I}| = \lambda^2 - 1 \to (\lambda - 1)(\lambda + 1) \to \lambda_1 = 1 \quad \lambda_2 = -1$$

for $\lambda = 1$ $\quad x = [1, 1]^T$ from problem #3 $\quad \to$ General Solution : $y' = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t}$

for $\lambda = -1$ $\quad x = [1, -1]^T$ from problem #3

$$y = \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solve for Particular solution

$$\vec{y}^{(P)} = \vec{Y}(t)\vec{u}(t)$$

$\vec{A}\vec{Y}\vec{u} + \vec{g} = \vec{Y}'\vec{u} + \vec{Y}\vec{u}'$, from problem #3 we know $\vec{u}' = \vec{Y}^{-1}\vec{g}$

$$\vec{Y}^{-1} = \frac{1}{\det(A)}\begin{vmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{vmatrix} \to \frac{1}{-2e^{0}}\begin{bmatrix} -e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{bmatrix}$$

$$\vec{u}' = -\frac{1}{2}\begin{bmatrix} -e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{bmatrix}\begin{bmatrix} 6e^{2t} \\ -e^{2t} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -6e^{t} + e^{t} \\ -6e^{3t} - e^{3t} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -5e^{t} \\ -7e^{3t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5e^{t} \\ 7e^{3t} \end{bmatrix}$$

$$\vec{u} = \int_0^t \frac{1}{2}\begin{bmatrix} 5e^{t} \\ 7e^{3t} \end{bmatrix} dt = \begin{matrix} \frac{1}{2}5e^{t}\big|_0^t \\ \frac{1}{6}7e^{3t}\big|_0^t \end{matrix} \to \begin{matrix} \frac{1}{2}5e^{t} - 5/2 \\ \frac{1}{6}7e^{3t} - 7/6 \end{matrix} \to \frac{1}{2}\begin{bmatrix} 5e^{t} \\ \frac{7}{3}e^{3t} \end{bmatrix} - \begin{bmatrix} 5/2 \\ 7/6 \end{bmatrix}$$

$$y^{(P)} = Y(t)u(t) = \frac{1}{2}\begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 5e^{t} \\ \frac{7}{3}e^{3t} \end{bmatrix} + -\begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}\begin{bmatrix} 5/2 \\ 7/6 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 5e^{2t} + \frac{7}{3}e^{2t} \\ 5e^{2t} - \frac{7}{3}e^{2t} \end{bmatrix} - \frac{5}{2}e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{7}{6}e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{22}{3}e^{2t} \\ \frac{8}{3}e^{2t} \end{bmatrix} - \frac{5}{2}e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{7}{6}e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 11/3 \\ 4/3 \end{bmatrix}e^{2t} + -\frac{5}{2}e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + -\frac{7}{6}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 11 \\ 4 \end{bmatrix}e^{2t} - \frac{5}{2}e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{7}{6}e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Combine with general solution

Notice that the last two components match components in the general solution therefore they are absorbed.

$$y(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + \frac{1}{3}\begin{bmatrix} 11 \\ 4 \end{bmatrix}e^{2t}$$

Solve for initial conditions

$$y_1(0) = c_1 e^{0} + c_2 e^{0} + \frac{1}{3}(11)e^{0} \quad \to \quad c_1 + c_2 + 11/3 = 1$$

$$y_2(0) = c_1 e^{0} + -c_2 e^{0} + \frac{1}{3}(4)e^{0} \qquad c_1 - c_2 + 4/3 = 0$$

$$2c_1 + 15/3 = 1 \to c_1 = -2$$

$$c_2 = 1 - 11/3 + 2 = -2/3$$

Combine with solution

$$\boxed{y(t) = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t} + -\frac{2}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + \frac{1}{3}\begin{bmatrix} 11 \\ 4 \end{bmatrix}e^{2t}}$$