Homework 5 for MA527 by Yonyi Bi (PUID: 0025680690)

40/40.

§4.4 P151

3. ~~Answer~~:

obviously, $\vec{y}' = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \vec{y}$, $\det \begin{vmatrix} -\lambda & 1 \\ -9 & -\lambda \end{vmatrix} = \lambda^2 + 9 = 0$

$\Rightarrow \lambda_1 = 3i, \quad \lambda_2 = -3i$

Because λ_1, λ_2 are pure imaginary, $p = \lambda_1 + \lambda_2 = 0$, $q = \lambda_1\lambda_2 > 0$

hence the critical point is the type of center, and stable.

For $\lambda_1 = 3i$, $-(3i)x_1 + x_2 = 0 \Rightarrow \vec{x_1} = [1 \;\; 3i]^T$

For $\lambda_2 = -3i$, $-(-3i)x_1 + x_2 = 0 \Rightarrow \vec{x_2} = [1 \;\; -3i]^T$

hence

$$\vec{y} = c_1 \begin{bmatrix} 1 \\ 3i \end{bmatrix} e^{3it} + c_2 \begin{bmatrix} 1 \\ -3i \end{bmatrix} e^{-3it}$$

$$\Rightarrow \begin{cases} y_1 = c_1 e^{3it} + c_2 e^{-3it} \\ y_2 = 3c_1 i e^{3it} - 3c_2 i e^{-3it} \end{cases}$$

, According to Euler Formula $e^{ix} = \cos x + i \sin x$

we obtain

$y_1 = c_1(\cos 3t + i \sin 3t) + c_2(\cos 3t - i \sin 3t)$

$= (c_1 + c_2)\cos 3t + (c_1 - c_2) i \sin 3t$, let $A = c_1 + c_2$, $B = (c_1 - c_2)i$

then $y_1 = A \cos 3t + B \sin 3t$

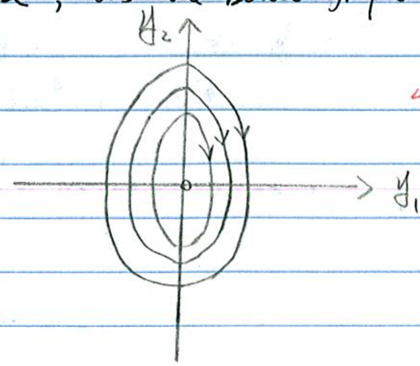
and $y_2 = 3c_1 i(\cos 3t + i \sin 3t) - 3c_2 i(\cos 3t - i \sin 3t)$

$= (3c_1 i - 3c_2 i)\cos 3t + (-3c_1 - 3c_2)\sin 3t$

$= 3B \cos 3t - 3A \sin 3t$, and clearly, $y_1^2 + (\frac{y_2}{3})^2 = \text{const}$

To determine the direction of the graph, $\vec{y}'$ at $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \end{bmatrix}$

hence, it's clockwise, as the below graph showing:



5. Answer:

$$\vec{y}' = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \vec{y}, \quad \det \begin{vmatrix} -2-\lambda & 2 \\ -2 & -2-\lambda \end{vmatrix} = (\lambda+2)^2+4 = \lambda^2+4\lambda+8=0$$

$$\Rightarrow \lambda_1 = -2+2i, \quad \lambda_2 = -2-2i$$

Because $p=\lambda_1+\lambda_2 = -4<0$, $q=\lambda_1\lambda_2 = 4+4>0$

hence the type of critical point is spiral, stable and attractive

For $\lambda_1 = -2+2i$, $(-2-(-2+2i))x_1+2x_2=0 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

For $\lambda_2 = -2-2i$, $(-2-(-2-2i))x_1+2x_2=0 \Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

hence $\vec{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-2+2i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-2-2i)t}$

$$= c_1 e^{-2t} \begin{bmatrix} e^{2it} \\ i e^{2it} \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} e^{-2it} \\ -i \cdot e^{-2it} \end{bmatrix}$$

$$\Rightarrow \begin{cases} y_1 = e^{-2t}(c_1 e^{2it} + c_2 e^{-2it}) \\ y_2 = e^{-2t}(c_1 i e^{2it} - c_2 i e^{-2it}) \end{cases}$$

According to Euler Formula, $e^{ix} = \cos x + i\sin x$

$\Rightarrow y_1 = e^{-2t}((c_1+c_2)\cos 2t + (c_1-c_2)i\sin 2t)$ Let $A = c_1+c_2$, $B = (c_1-c_2)i$

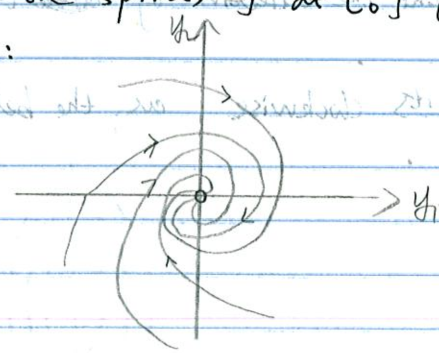
then $y_1 = e^{-2t}(A\cos 2t + B\sin 2t)$

$y_2 = e^{-2t}((c_1-c_2)i\cos 2t - (c_1+c_2)\sin 2t)$

$= e^{-2t}(B\cos 2t - A\sin 2t)$

To determine the direction of the spiral, $\vec{y}'$ at $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

then the graph is as below:



7. Answer:

$$\vec{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{y}, \quad \det \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0$$

$\Rightarrow \lambda_1 = 3; \ \lambda_2 = -1$

Because $p = \lambda_1 + \lambda_2 = 2 > 0, \ q = \lambda_1 \lambda_2 < 0$

hence The type of the critical point is saddle and unstable.

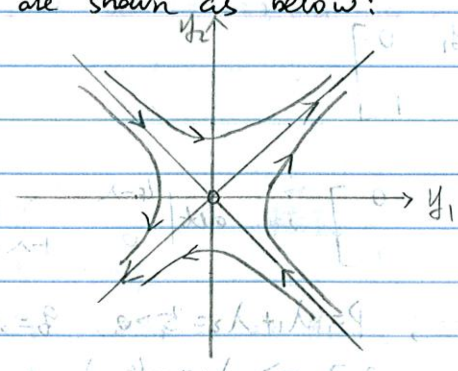
For $\lambda_1 = 3$, $-2x_1 + 2x_2 = 0 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = -1$, $2x_1 + 2x_2 = 0 \Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

hence $\vec{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

$$\Rightarrow \begin{cases} y_1 = c_1 e^{3t} + c_2 e^{-t} \\ y_2 = c_1 e^{3t} - c_2 e^{-t} \end{cases}$$

The trajectories are shown as below:



11. Answer:

$$\begin{cases} y_1 = y \\ y_2 = y_1' = y' \\ y_3 = y_2'' = y'' = -2y' - 2y \end{cases} \Rightarrow \vec{y}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \vec{y}$$

$$\det \begin{vmatrix} 0-\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda_1 = -1+i, \ \lambda_2 = -1-i$$

Because $p = \lambda_1 + \lambda_2 = -2 < 0$, $q = \lambda_1 \lambda_2 = 2 > 0$

hence, the trajectories are spirals and stable, and attractive.

For $\lambda_1 = -1+i$, $(1-i)x_1 + x_2 = 0 \Rightarrow \vec{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For $\lambda_2 = -1-i$, $(1+i)x_1 + x_2 = 0 \Rightarrow \vec{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$y = y_1 = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t}$

$= e^{-t}\left[(c_1+c_2)\cos t + (c_1-c_2) i \sin t \right]$

Let $A = c_1 + c_2$, $B = (c_1 - c_2) i$

$\Rightarrow y = e^{-t}(A\cos t + B\sin t)$

§4.5 P159.

4. Answer:

$f_1 = 4y_1 - y_1^2 = y_1(4-y_1) = 0 \Rightarrow y_1 = 0$ or $y_1 = 4$

$f_2 = y_2 = 0$

hence we have two critical points $(0,0)$ and $(4,0)$

And $\nabla \vec{f} = \begin{bmatrix} 4-2y_1 & 0 \\ 0 & 1 \end{bmatrix}$

$\nabla \vec{f}(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$, $\det \begin{vmatrix} 4-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (\lambda-1)(\lambda-4) = 0$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 4$, $p = \lambda_1 + \lambda_2 = 5 > 0$, $q = \lambda_1 \lambda_2 = 4 > 0$

$\nabla \vec{f}(4,0) = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \lambda_1 = -4, \lambda_2 = 1$

$p = \lambda_1 + \lambda_2 = -3 < 0$, $q = \lambda_1 \lambda_2 = -4 < 0$

Hence, proper node at $(0,0)$, unstable

saddle point at $(4,0)$, unstable

7. Answer:

$f_1 = -y_1 + y_2 - y_2^2 = 0$, $f_2 = -y_1 - y_2 = 0$

$\Rightarrow \quad 2y_2 - y_2^2 = 0 \Rightarrow y_2 = 0$ or $y_2 = 2$

then we have two critical points $(0,0)$, $(-2,2)$

And $\vec{\nabla f} = \begin{bmatrix} -1 & 1-2y_2 \\ -1 & -1 \end{bmatrix}$

$$\vec{\nabla f}(0,0) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \det \begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0$$

$$\lambda_1 = -1+i, \lambda_2 = -1-i, \quad p = \lambda_1 + \lambda_2 = -2 < 0, \quad q = \lambda_1 \lambda_2 = 2 > 0$$

$$\vec{\nabla f}(-2,2) = \begin{bmatrix} -1 & -3 \\ -1 & -1 \end{bmatrix}, \det \begin{vmatrix} -1-\lambda & -3 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 3 = 0$$

$$\lambda_1 = 1, \lambda_2 = -3, \quad p = \lambda_1 + \lambda_2 = -2 < 0, \quad q = \lambda_1 \lambda_2 = -3 < 0$$

Hence, spiral point at $(0,0)$, stable and attractive

saddle point at $(-2,2)$, unstable

11. Answer:

$y_1 = y$

$y_2 = y_1' = y'$ $\quad \Rightarrow \quad \cancel{\vec{y}' = \begin{bmatrix} 0 & 1 \\ & \end{bmatrix} \vec{y}}$ $\quad \begin{cases} y_1' = y_2 \\ y_2' = -\cos y_1 \end{cases}$

$y_3 = y_2' = y'' = -\cos y$

$f_1 = y_2 = 0, \; f_2 = -\cos y_1 = 0 \Rightarrow y_1 = \frac{\pi}{2} \pm 2n\pi$, or $y_1 = -\frac{\pi}{2} \pm 2n\pi$

then we have two sets of critical points:

$(\frac{\pi}{2} \pm 2n\pi, 0)$ and $(-\frac{\pi}{2} \pm 2n\pi, 0)$

And $\vec{\nabla f} = \begin{bmatrix} 0 & 1 \\ \sin y_1 & 0 \end{bmatrix}$

$$\vec{\nabla f}(\tfrac{\pi}{2} \pm 2n\pi, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \det \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$\Rightarrow \lambda_1 = 1, \lambda_2 = -1, \quad p = \lambda_1 + \lambda_2 = 0, \quad q = \lambda_1 \lambda_2 = -1 < 0$

$\vec{\nabla f}(-\frac{\pi}{2}\pm 2n\pi, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\det\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2+1=0$

$\Rightarrow \lambda_1 = i, \quad \lambda_2 = -i$

$p = \lambda_1+\lambda_2 = 0, \quad q = \lambda_1\lambda_2 = 1 > 0$

Hence, Saddle points at $(\frac{\pi}{2}\pm 2n\pi, 0)$, unstable

centers at $(-\frac{\pi}{2}\pm 2n\pi, 0)$, stable

§4.6 P163.

3. Answer:

$$\vec{y}\,' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{y} + \begin{bmatrix} 1 \\ -3 \end{bmatrix}e^{3t}$$

For the homogeneous system,

$$\det\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2-1=0 \Rightarrow \lambda_1=1,\ \lambda_2=-1$$

$\lambda_1 = 1, \quad -x_1+x_2=0 \Rightarrow \vec{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2=-1, \quad x_1+x_2=0 \Rightarrow \vec{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

hence, $\vec{y_h} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t}$

Let $\vec{y_p} = e^{3t}\vec{v} + te^{3t}\vec{u}$

$$\begin{aligned}\vec{y_p}\,' &= 3e^{3t}\vec{v} + (1+3t)e^{3t}\vec{u} \\ &= A(e^{3t}\vec{v} + t\,e^{3t}\vec{u}) + \begin{bmatrix} 1 \\ -3 \end{bmatrix}e^{3t}\end{aligned}$$

$\Rightarrow 3\vec{v} + (1+3t)\vec{u} = A\vec{v} + tA\vec{u} + \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$\Rightarrow t(A-3I)\vec{u} = \vec{0}$, then $\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\vec{u} = \vec{0} \Rightarrow \vec{u} = \vec{0}$

and $\quad (A-3I)\vec{v} = \vec{u} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

then $\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$\Rightarrow \begin{cases} -3v_1 + v_2 = -1 \\ v_1 - 3v_2 = 3 \end{cases} \Rightarrow \begin{cases} v_1 = 0 \\ v_2 = -1 \end{cases} \quad \vec{v} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Hence $\vec{y} = \vec{y_h} + \vec{y_p} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{3t}$

$$\Rightarrow \begin{cases} y_1 = c_1 e^{t} + c_2 e^{-t} \\ y_2 = c_1 e^{t} - c_2 e^{-t} - e^{3t} \end{cases}$$

11. Answer:

$$\vec{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{y} + \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{2t}$$

For the homogeneous system, $\det \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0$, $\lambda_1 = -1$, $\lambda_2 = 1$

$\lambda_1 = -1$, $\quad x_1 + x_2 = 0 \Rightarrow \vec{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\lambda_2 = 1$, $-x_1 + x_2 = 0 \Rightarrow \vec{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

hence $\vec{y_h} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$

Let $\quad \vec{y_p} = e^{2t} \vec{v} + t e^{2t} \vec{u}$

$$\vec{y_p}' = 2e^{2t} \vec{v} + (1+2t) e^{2t} \vec{u}$$
$$= A(e^{2t} \vec{v} + t e^{2t} \vec{u}) + \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{2t}$$

$$\Rightarrow 2\vec{v} + (1+2t)\vec{u} = A\vec{v} + tA\vec{u} + \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$\Rightarrow t(A-2I)\vec{u} = \vec{0}, \text{ then } \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{u} = \vec{0} \Rightarrow \vec{u} = \vec{0}$$

and $(A-2I)\vec{v} = \vec{u} - \begin{bmatrix} 6 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

$$\begin{cases} -2v_1 + v_2 = -6 \\ v_1 - 2v_2 = 1 \end{cases} \Rightarrow \begin{cases} v_1 = \frac{11}{3} \\ v_2 = \frac{4}{3} \end{cases}$$

Hence $\vec{y} = \vec{y_h} + \vec{y_p}$

$$= c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} \frac{11}{3} \\ \frac{4}{3} \end{bmatrix} e^{2t}$$

$$\Rightarrow y_1 = c_1 e^{-t} + c_2 e^{t} + \frac{11}{3} e^{2t}$$

$$y_2 = -c_1 e^{-t} + c_2 e^{t} + \frac{4}{3} e^{2t}$$

And $y_1(0) = 1$, $y_2(0) = 0$

$$\Rightarrow \begin{matrix} c_1 + c_2 + \frac{11}{3} = 1 \\ -c_1 + c_2 + \frac{4}{3} = 0 \end{matrix} \Rightarrow \begin{cases} c_1 = -\frac{2}{3} \\ c_2 = -2 \end{cases}$$

$$y_1 = -\frac{2}{3} e^{-t} - 2e^{t} + \frac{11}{3} e^{2t}$$

$$y_2 = \frac{2}{3} e^{-t} - 2e^{t} + \frac{4}{3} e^{2t}$$

Because $e^{t} = \sinh t + \cosh t$, $e^{-t} = \cosh t - \sinh t$, it can alse be written as:

$$y_1 = -\frac{2}{3}(\cosh t - \sinh t) - 2(\cosh t + \sinh t) + \frac{11}{3} e^{2t}$$

$$= -\frac{8}{3}\cosh t - \frac{4}{3}\sinh t + \frac{11}{3} e^{2t}$$

$$y_2 = \frac{2}{3}(\cosh t - \sinh t) - 2(\cosh t + \sinh t) + \frac{4}{3} e^{2t}$$

$$= -\frac{4}{3}\cosh t - \frac{8}{3}\sinh t + \frac{4}{3} e^{2t}$$

Mead