$\boxed{6.1-1}$ ✓

MA527
HW #6
Jay Guthrie

$$3t + 12 = f(t)$$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st}(3t+12)\,dt = \underbrace{3\int_0^\infty te^{-st}\,dt} + 12\int_0^\infty e^{-st}\,dt$$

$u = t \quad dv = e^{-st}dt$

$$\int te^{-st}\,dt = t\left(-\frac{1}{s}e^{-st}\right) - \int\left(-\frac{1}{s}e^{-st}\right)dt$$

$$= -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}$$

$$\mathcal{L}(f) = \frac{-3(ts+1)}{s^2}e^{-st} - \frac{12}{s}e^{-st}\Big|_0^\infty = \boxed{\frac{3}{s^2} + \frac{12}{s}}$$

Or using table...

$$\mathcal{L}(3t+12) = \frac{3}{s^2} + \frac{12}{s}$$

$\boxed{6.1-2}$ ✓

$$f(t) = (a - bt)^2$$

$$= a^2 - 2abt + b^2t^2 \qquad (a \text{ and } b \text{ are constants})$$

Using table...

$$\mathcal{L}(f) = \frac{a^2}{s} - \frac{2ab}{s^2} + \frac{2b^2}{s^3}$$

Integral

$$\int_0^\infty a^2 e^{-st}\,dt - \int_0^\infty 2abte^{-st}\,dt + b^2\int_0^\infty t^2 e^{-st}\,dt$$

$$u = t^2 \quad dv = e^{-st}\,dt$$

$$\int_0^\infty t^2 e^{-st}\,dt = \cancel{\frac{-1}{s}t^2 e^{-st}\Big|_0^\infty} \to 0 + \int_0^\infty \frac{1}{s}e^{-st}\,2t\,dt = \frac{2}{s}\underbrace{\int_0^\infty e^{-st}\,t\,dt} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}$$

Previously showed this evaluates to $\frac{1}{s^2}$ with $\Big|_0^\infty$

$$\boxed{\frac{a^2}{s} - \frac{2ab}{s^2} + \frac{2b^2}{s^3}}$$

$\boxed{6.1\text{-}5}$ ✓

$$f(t) = e^{2t} \sinh t \qquad \sinh t = \frac{1}{2}\left(e^{t} - e^{-t}\right)$$

$$\mathcal{L}(f) = \int_0^\infty e^{-st} e^{2t} \frac{1}{2}\left(e^{t} - e^{-t}\right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-2-1)t}\, dt - \frac{1}{2}\int_0^\infty e^{-(s-2+1)t}\, dt$$

$$= \frac{1}{2}\left(\frac{-1}{s-3}\right) e^{-(s-3)t}\Big|_0^\infty - \frac{1}{2}\left(\frac{-1}{s-1}\right) e^{-(s-1)t}\Big|_0^\infty$$

$$= \boxed{\frac{1}{2(s-3)} - \frac{1}{2(s-1)}} \quad \text{or equivalently} \quad \frac{2s-2-2s+6}{4s^2-16s+12} = \frac{1}{s^2-4s+3} = \boxed{\frac{1}{(s-2)^2-1}}$$

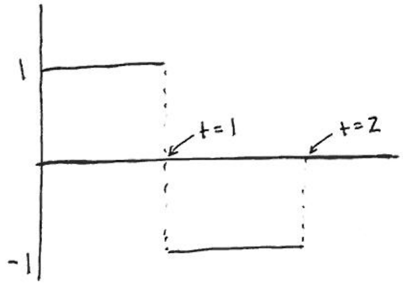
Using table...

$$f(t) = \frac{1}{2}e^{3t} - \frac{1}{2}e^{t} \qquad \mathcal{L}(f) = \frac{1}{2}\left(\frac{1}{s-3}\right) - \frac{1}{2}\left(\frac{1}{s-1}\right)$$

even simpler from $\sinh at \Rightarrow \frac{a}{s^2-a^2}$ w/ shift of 2 due to e^{2t}

$$\mathcal{L}(f) = \frac{1}{(s-2)^2-1}$$

6.1-13



After $t=2$, $f(t)=0$ so $\mathcal{L}(f)$ is zero after $t=2$

$$f(t)=\begin{cases}1 & 0<t<1\\ -1 & 1<t<2\end{cases}$$

$$\mathcal{L}(f)=\int_0^1 e^{-st}(1)\,dt+\int_1^2 e^{-st}(-1)\,dt$$

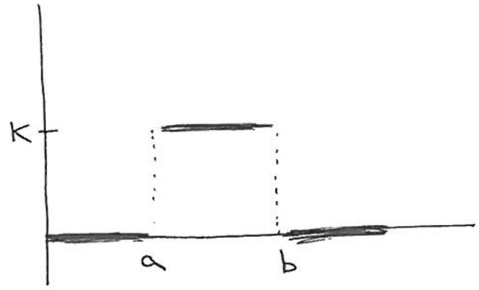
$$=\frac{-1}{s}e^{-st}\Big|_0^1+\frac{1}{s}e^{-st}\Big|_1^2$$

$$=\frac{-1}{s}\left(e^{-s}-1\right)+\frac{1}{s}\left(e^{-2s}-e^{-s}\right)$$

$$=\frac{1}{s}\left(1+e^{-2s}-2e^{-s}\right)$$

$$=\boxed{\frac{1}{s}\left(1-e^{-s}\right)^2}$$

$\boxed{6.1-14}$ ✓



$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \cancel{\int_0^a e^{-st} \cdot 0\,dt} + \int_a^b K e^{-st}\,dt + \cancel{\int_b^\infty e^{-st}\, 0\,dt}$$

$$= -\frac{1}{s} K e^{-st} \Big|_a^b = \frac{1}{s} K \left(e^{-as} - e^{-bs}\right)$$

$$\boxed{F(s) = \frac{1}{s} K \left(e^{-as} - e^{-bs}\right)}$$

$\boxed{6.1-23}$ *

If $\mathcal{L}(f(t)) = F(s)$ and c is any positive constant,

show that $\mathcal{L}(f(ct)) = \dfrac{F(\frac{s}{c})}{c}$

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}(f(ct)) = \int_0^\infty e^{-st} f(ct)\, dt$$

let $ct = x$

$d(ct) = dx$

$c\,dt = dx$

$$\mathcal{L}(f(x)) = \int_0^\infty e^{-s\frac{x}{c}} \frac{f(x)}{c}\, dx = \frac{1}{c}\int e^{-\frac{s}{c}x} f(x)\, dx$$

$$= \frac{F(\frac{s}{c})}{c}$$

$$\mathcal{L}(\cos t) = \frac{s}{s^2 + 1^2}$$

$$\mathcal{L}(\cos wt) = \frac{F(\frac{s}{w})}{w} = \frac{s/w}{(s/w)^2 + 1^2} \cdot \frac{1}{w} = \frac{s}{s^2 + w^2}$$

$\boxed{6.1\text{-}30}$ ✓

$$\mathcal{L}(f) = \frac{4s+32}{s^2-16}$$

$$= 4\,\frac{s}{s^2-4^2} + 8\,\frac{4}{s^2-4^2}$$

$$\boxed{f(t) = 4\cosh(4t) + 8\sinh(4t)}$$

$\boxed{6.1\text{-}32}$ ✓

$$\mathcal{L}(f) = \frac{1}{(s+a)(s+b)} = \frac{1}{s^2 + (a+b)s + ab}$$

PFE

$$= \frac{1}{s+a}\cdot\frac{1}{(b-a)} + \frac{1}{s+b}\cdot\frac{1}{(a-b)}$$

Assuming $a \neq b$

$$\boxed{f(t) = \frac{1}{b-a}e^{-at} + \frac{1}{a-b}e^{-bt}}$$

If $a = b$

$$\mathcal{L}(f) = \frac{1}{(s+a)^2}$$ shifting $$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s+a)^2}\right) = e^{-at}t$$

$\boxed{6.2-4}$ ✓

$$y'' + 9y = 10e^{-t} \qquad y(0) = 0 \qquad y'(0) = 0$$

$$s^2Y - \cancel{sy(0)} - \cancel{y'(0)} + 9Y = \frac{10}{s+1}$$

$$Y = \frac{1}{s^2+9} \cdot \frac{10}{s+1}$$

PFE

$$= \frac{As + B}{s^2+9} + \frac{C}{s+1}$$

$$\cancel{As^2 + B + Cs^2 + 9C = 10}$$
$$\cancel{A + C = 0}$$
$$\cancel{B + 9C = 0}$$

$$As^2 + Bs + As + B + Cs^2 + 9C = 10$$

$$A + C = 0 \qquad A + B = 0 \;\rightarrow\; C = B \;\Rightarrow\; C = 1,\ A = -1,\ B = 1$$
$$B + 9C = 10$$

$$= \frac{-s+1}{s^2+9} + \frac{1}{s+1} = -\frac{s}{s^2+3^2} + \frac{1}{3}\cdot\frac{3}{s^2+3^2} + \frac{1}{s+1}$$

$$\mathcal{L}^{-1}(F(s)) = \boxed{-\cos(3t) + \frac{1}{3}\sin(3t) + e^{-t}}$$

$\boxed{6.2-5}$ ✓

$$y'' - \frac{1}{4}y = 0 \qquad y(0) = 12 \qquad y'(0) = 0$$

$$\left[s^2Y - sy(0) - y'(0)\right] + \frac{-1}{4}Y = 0$$

$$\left(s^2 - \frac{1}{4}\right)Y = 12s$$

$$Y = \frac{12s}{s^2 - \frac{1}{4}} = 12\left(\frac{s}{s^2 - \left(\frac{1}{2}\right)^2}\right)$$

$$y(t) = \mathcal{L}^{-1}(Y) = \boxed{12\cosh\left(\tfrac{1}{2}t\right)}$$

or equiv. $\frac{12}{2}\left(e^{t/2} + e^{-t/2}\right)$

grows exponentially w/ time

$\boxed{6.2\text{-}17}$ ✓

$f(t) = te^{-at}$

$f'(t) = e^{-at} - tae^{-at}$ $\qquad f(0) = 0$

$\mathcal{L}(f') = s\mathcal{L}(f) - \cancel{f(0)}$

$\mathcal{L}(e^{-at}) - a\mathcal{L}(te^{-at}) = s\mathcal{L}(te^{-at})$

$\mathcal{L}(e^{-at}) = (s+a)\mathcal{L}(te^{-at})$

$$\mathcal{L}(te^{-at}) = \frac{1}{s+a}\mathcal{L}(e^{-at})$$

$$= \frac{1}{s+a} \cdot \frac{1}{s+a}$$

$$= \boxed{\frac{1}{(s+a)^2}}$$

$\boxed{6.2\text{-}19}$ ✓

$f(t) = \sin^2 \omega t$

$f'(t) = 2\omega(\sin \omega t)(\cos \omega t)$ trig id: $\sin 2\theta = 2\sin\theta\cos\theta$

$= \omega \sin 2\omega t$

$f(0) = 0$

$\mathcal{L}(f') = s\mathcal{L}(f) - \cancel{f(0)}$

$\omega \dfrac{(2\omega)}{s^2 + (2\omega)^2} = s\mathcal{L}(f)$

$\mathcal{L}(f) = \boxed{\dfrac{2\omega^2}{s(s^2+4\omega^2)}}$

$\boxed{6.2-26}$ ✓

$$\mathcal{L}(f) = \frac{1}{s^4 - s^2}$$

$$\mathcal{L}\left\{ \int_0^t f(\tau)\, d\tau \right\} = \frac{1}{s} F(s) \qquad \int_0^t f(\tau)\, d\tau = \mathcal{L}^{-1}\left\{ \frac{1}{s} F(s) \right\}$$

$$\mathcal{L}(f) = \frac{1}{s^2(s^2-1)}$$ Double integral

$$\mathcal{L}^{-1}\left(\frac{1}{s^2} \cdot \frac{1}{s^2-1} \right) = \int_0^t \int_0^t f(\tau)\, d\tau\, d\tau \qquad f(\tau) = \sinh t$$

$$= \int_0^t \int_0^t \sinh \tau \, d\tau\, d\tau = \int_0^t \cosh \tau \, d\tau = \sinh t$$

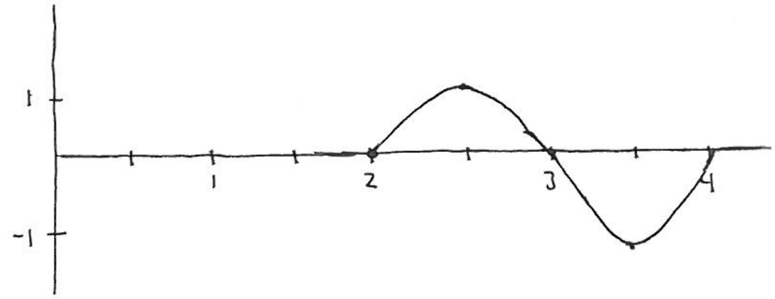
redo integral

$$\cosh \tau \Big|_0^t = \cosh t - 1$$

$$\int_0^t (\cosh \tau - 1)\, d\tau = \sinh \tau \Big|_0^t - t = \boxed{\sinh t - t}$$

$\boxed{6.3-6}$

$f(t) = \sin \pi t \quad (2 < t < 4)$



$$f(t) = \sin \pi t \left[u(t-2) - u(t-4) \right]$$

$$\mathcal{L}\left\{ \sin \pi t \left[u(t-2) \right] \right\} + \mathcal{L}\left\{ \sin \pi t \left[-u(t-4) \right] \right\}$$

$$= e^{-2s} \mathcal{L}\{ \underbrace{\sin \pi(t+2)}_{\pi t + 2\pi} \} + -e^{-4s} \mathcal{L}\{ \underbrace{\sin \pi(t+4)}_{\pi t + 4\pi} \}$$

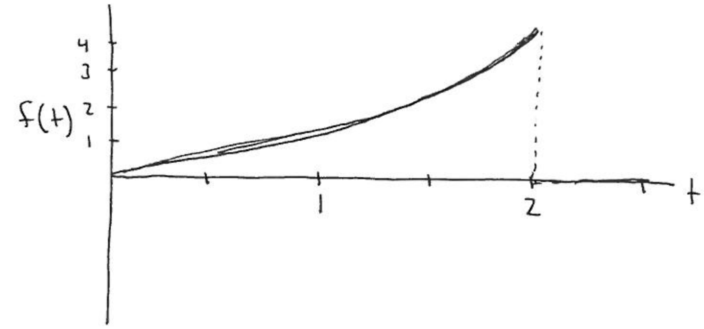
trig. id:
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$$= e^{-2s} \mathcal{L}\{ \sin \pi t \} + -e^{-4s} \mathcal{L}\{ \sin \pi t \}$$

$$= \boxed{e^{-2s} \frac{\pi}{s^2 + \pi^2} - e^{-4s} \frac{\pi}{s^2 + \pi^2}}$$

$\boxed{6.3-10}$ ✓ $f(t) = \sinh t \quad 0 < t < 2$

$= 0 \quad 2 < t$



$f(t) = \sinh t - (\sinh t)\,u(t-2)$

$\sinh(x+y) = \cosh(x)\sinh(y) + \sinh(x)\cosh(y)$

$\mathcal{L}(f) = \dfrac{1}{s^2-1^2} - e^{-2s}\,\mathcal{L}\left(\sinh(t+2)\right)$

$= \dfrac{1}{s^2-1} - e^{-2s}\,\mathcal{L}\left(3.6269\cosh t + 3.7622\sinh t\right)$

$$= \boxed{\frac{1}{s^2-1} - e^{-2s}\left[3.6269\frac{s}{s^2-1} + 3.7622\frac{1}{s^2-1}\right]}$$

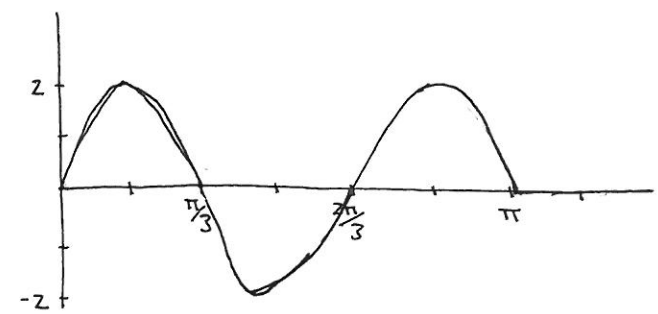
$\boxed{6.3-13}$

$$\mathcal{L}(f) = \frac{6(1-e^{-\pi s})}{s^2+9}$$

$$= 2\,\frac{3}{s^2+3^2}\,(1-e^{-\pi s})$$

$$\mathcal{L}^{-1}(\ ) = 2\sin 3t \quad + {}^{-}2\sin(3t-\pi)\,u(t-\pi)$$

$$= \boxed{2\sin 3t\,(1-u(t-\pi))}$$

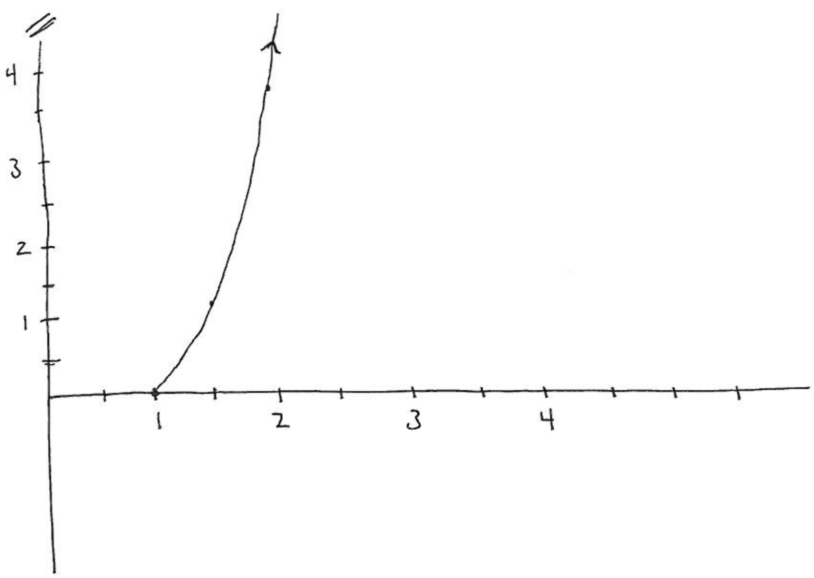


$\boxed{6.3-16}$ ✓

$$\mathcal{L}(f) = \frac{2(e^{-s} - e^{-3s})}{s^2 - 4}$$

$$= \frac{2}{s^2 - 2^2} e^{-s} - \frac{2}{s^2 - 2^2} e^{-3s}$$

$$\boxed{f(t) = \sinh(2(t-1))\,u(t-1) - \sinh(2(t-3))\,u(t-3)}$$



grows exponentially
2nd term is "behind" first term by $t=2$ so it never cancels out first term fully.

$\boxed{6.3-25}$

$y'' + y = t$ if $0<t<1$ $\quad y(0)=0 \quad y'(0)=0$

$\ 0$ if $1<t$

$y'' + y = t(1 - u(t-1))$

p.214

$\left[s^2 Y - s y(0) - y'(0)\right] + Y = \frac{1}{s} - \frac{e^{-s}}{s}$

$\left[s^2 + 1\right] Y = \frac{1}{s} - \frac{e^{-s}}{s}$

$Y = \frac{1}{s(s^2+1)} - \frac{e^{-s}}{s(s^2+1)}$

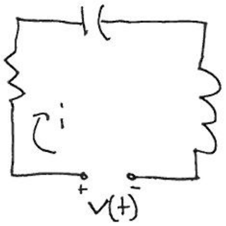
PFE

$= \frac{1}{s} - \frac{s}{s^2+1} + - \frac{1}{s} e^{-s} + \frac{s}{s^2+1} e^{-s}$

$\mathcal{L}^{-1}(\ ) = \boxed{t - \cos t - (t-1)u(t-1) + \cos(t-1)u(t-1)}$

$\boxed{6.3\text{-}39}$ ✓

$i(0) = 0 \qquad Q(0) = 0$



$R = 2\ \Omega$

$L = 1\ H$

$C = 0.5\ F$

$v(t) = \begin{cases} 1\ kV & 0 < t < 2 \\ 0 & 2 < t \end{cases}$

$$v(t) = Ri + \frac{1}{C}\int_0^t i\, d\xi + L\frac{di}{dt}$$

$$K(1 - u(t-2)) =$$

$$K\left(\frac{1}{s} - \frac{e^{-2s}}{s}\right) = I\left(R + \frac{1}{sC} + sL\right) = I\left(2 + \frac{2}{s} + s\right)$$

$$I = 1000\frac{1}{2s + 2 + s^2} - 1000\frac{e^{-2s}}{2s + 2 + s^2}$$

$$I = 1000\frac{1}{(s+1)^2 + 1} - 1000\frac{e^{-2s}}{(s+1)^2 + 1}$$

Inverse L.T.

$$\boxed{i(t) = 1000\left(e^{-t}\sin t - e^{-(t-2)}\sin(t-2)\,u(t-2)\right)}$$