Dan Lovelace   MA 527   1

p230 # 3, 10, 14ab

3. $y'' + 4y = \delta(t-\pi) \quad y(0) = 8 \quad y'(0) = 0$

$\mathcal{L} \quad s^2Y - sy(0) - y'(0) + 4Y = e^{-\pi s}$

$Y(s^2+4) - 8s = e^{-\pi s} \qquad Y = \frac{e^{-\pi s} + 8s}{s^2+4} = \frac{e^{-\pi s}}{s^2+2^2} + \frac{8 \cdot s}{s^2+2^2}$

$\mathcal{L}^{-1} \quad y = \mathcal{L}^{-1}\left(e^{-\pi s}\frac{1}{s^2+2^2}\right) + 8\mathcal{L}^{-1}\left(\frac{s}{s^2+2^2}\right)$

$= \frac{1}{2}\sin 2(t-\pi)\,u(t-\pi) + 8\cos 2t$

$= \frac{1}{2}\sin(2t-2\pi)\,u(t-\pi) + 8\cos 2t$

$\sin(2t-2\pi) = \sin(2t)$

$$\boxed{y = \frac{1}{2}\sin(2t)\,u(t-\pi) + 8\cos(2t)}$$

told not to graph by professor

10. $y'' + 5y' + 6y = \delta(t-\frac{\pi}{2}) + u(t-\pi)\cos t \qquad y(0)=0 \quad y'(0)=0$

$\mathcal{L} \quad s^2Y - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} + 5[sY - \cancel{y(0)}^{0}] + 6Y = e^{-\pi/2 s} + \mathcal{L}-\cos(t-\pi)u(t-\pi)$

$s^2Y + 5sY + 6Y = e^{-\pi/2 s} - e^{-\pi s}\mathcal{L}\cos t = e^{-\pi/2 s} - e^{-\pi s}\frac{s}{s^2+1}$

$Y(s^2+5s+6) = e^{-\frac{\pi}{2}s} - e^{-\pi s}\frac{s}{s^2+1}$

$Y = \frac{1}{(s+2)(s+3)}\left[e^{-\frac{\pi}{2}s} - e^{-\pi s}\frac{s}{s^2+1}\right]$

$\frac{1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3} \qquad 1 = A(s+3) + B(s+2)$
$A = 1 \quad B = -1$

$\frac{s}{(s+2)(s+3)(s^2+1)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1} \qquad s = A(s+3)(s^2+1) + B(s+2)(s^2+1) + (Cs+D)(s+2)(s+3)$

$s=-2 \quad -2 = 5A \quad A = -2/5 \qquad s=-3 \quad -3 = -10B \quad B = 3/10$
$s=0 \quad 0 = 3A + 2B + 6D \quad D = 1/10 \quad s=1 = 4A + 2B + (-C+D)2 \quad C = -1/10$

$y = \mathcal{L}^{-1}\left[e^{-\pi/2 s}\left(\frac{1}{s+2} - \frac{1}{s+3}\right)\right] - \mathcal{L}^{-1}\left[e^{-\pi s}\left(\frac{-2}{5}\frac{1}{s+2} + \frac{3}{10}\frac{1}{s+3} + \frac{1}{10}\frac{-s+1}{s^2+1}\right)\right]$

$$\boxed{y = \left[e^{-2(t-\pi/2)} - e^{-3(t-\pi/2)}\right]u(t-\pi/2) - \left[\frac{-2}{5}e^{-2(t-\pi)} + \frac{3}{10}e^{-3(t-\pi)} + \frac{-1}{10}\cos(t-\pi) + \frac{1}{10}\sin(t-\pi)\right]u(t-\pi)}$$

told not to graph by professor

14.ab.

a) prove $\mathcal{L}(f) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt \quad (s>0)$

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt \qquad \int_0^\infty \cdots = \int_0^p \cdots + \int_p^{2p} \cdots + \int_{2p}^{3p} \cdots \quad \cdots$$

$$\int_p^{2p} e^{-st} f(t)\,dt \qquad \begin{array}{c}\tau = t-p\\ d\tau = dt\end{array} \implies \int_0^p e^{-s(\tau+p)} f(\tau+p)\,d\tau$$

$$f(\tau+p) = f(\tau)$$

$$\int_0^p \left(e^{-s\tau} e^{-ps}\right) f(\tau)\,d\tau = e^{-ps} \int_0^p e^{-st} f(t)\,dt$$

$$\int_{2p}^{3p} e^{-st} f(t)\,dt \qquad \begin{array}{c}\tau = t-2p\\ d\tau = dt\end{array} \implies \int_0^p e^{-s(\tau+2p)} f(\tau+2p)\,d\tau = e^{-2ps} \int_0^p e^{-st} f(t)\,dt$$

$$\int_{3p}^{4p} \cdots \qquad \begin{array}{c}\tau = t-3p\\ d\tau = dt\end{array} \implies e^{-3ps} \int_0^p e^{-st} f(t)\,dt$$

$$\mathcal{L}(f) = \left[1 + e^{-ps} + e^{-2ps} + e^{-3ps} + \cdots\right] \int_0^p e^{-st} f(t)\,dt$$

let $x = 1 + e^{-ps} + (e^{-ps})^2 \cdots$ $\qquad x - e^{-ps}x = 1 \qquad x = \frac{1}{1-e^{-ps}}$

$e^{-ps}x = e^{-ps} + (e^{-ps})^2 \cdots$

$$\mathcal{L}(f) = x \int_0^p e^{-st} f(t)\,dt = \boxed{\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt}$$

b) $f = \sin \omega t$

$$\mathcal{L}(f) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt = \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} \sin\omega t\,dt = \cdots \int_0^{\pi/\omega} \cdots$$

$p = 2\pi/\omega$

$$\int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt \qquad \begin{array}{c}u = \sin\omega t\\ du = \omega\cos\omega t\end{array} \quad \begin{array}{c}dv = e^{-st}\\ v = -\frac{1}{s}e^{-st}\end{array} = \cancel{-\frac{\sin\omega t}{s} e^{-st}\Big|_0^{\pi/\omega}}^{\,0} - \int -\frac{\omega}{s} e^{-st} \cos\omega t$$

$$\int_0^{\pi/\omega} e^{-st}\sin\omega t = \frac{\omega}{s} \int_0^{\pi/\omega} e^{-st} \cos\omega t \qquad \begin{array}{c}u = \cos\omega t\\ du = -\omega\sin\omega t\end{array} \quad \begin{array}{c}dv = e^{-st}\\ v = -\frac{1}{s}e^{-st}\end{array} = \frac{\omega}{s}\left[-\frac{\cos\omega t}{s} e^{-st}\Big|_0^{\pi/\omega} - \frac{\omega}{s} \int_0^{\pi/\omega} e^{-st} \sin\omega t\right]$$

$$\int = \frac{\omega}{s}\left(\frac{1}{s}e^{-s\pi/\omega} + \frac{1}{s}\right) - \frac{\omega^2}{s^2}\int \qquad \int + \frac{\omega^2}{s^2}\int = \frac{\omega}{s^2}\left(e^{-s\pi/\omega} + 1\right)$$

$$\int = \frac{\frac{\omega}{s^2}\left(1 + e^{-s\pi/\omega}\right)}{1 + \omega^2/s^2} = \frac{\omega\left(1 + e^{-s\pi/\omega}\right)}{s^2 + \omega^2}$$

$$\frac{1}{1-e^{-2\pi s/\omega}}\left(\frac{\omega\left(1 + e^{-s\pi/\omega}\right)}{s^2+\omega^2}\right) = \boxed{\frac{\omega}{(s^2+\omega^2)\left(1 - e^{-\pi s/\omega}\right)}}$$

$u = \tau \quad dv = e^{\tau}$
$du = 1 \quad v = e^{\tau}$

p237 7, 8, 23

7. $t * e^t = \int_0^t (t-\tau)e^{\tau}d\tau = \int_0^t te^{\tau}d\tau - \int_0^t \tau e^{\tau}$

$te^{\tau}\Big|_0^t - \left[\tau e^{\tau}\Big|_0^t - \int_0^t e^{\tau}d\tau\right]$

$(\cancel{te^t} - t) - \left[(\cancel{te^t} - 0) - e^{\tau}\Big|_0^t\right]$

$-t + e^t - 1 = \boxed{e^t - t - 1}$

8. $y(t) + 4\int_0^t y(\tau)(t-\tau)d\tau = 2t$

$\mathcal{L} \quad Y + 4\mathcal{L}\underbrace{\int_0^t y(\tau)(t-\tau)d\tau}_{y * t} = 2/s^2$

$Y + 4\left(Y\frac{1}{s^2}\right) = \frac{2}{s^2} \qquad Y\left(1 + \frac{4}{s^2}\right) = \frac{2}{s^2}$

$Y = \frac{2}{s^2} \cdot \frac{1}{1 + 4/s^2} = \frac{2}{s^2+4} = \frac{2}{s^2+2^2}$

$\mathcal{L}^{-1} \quad \boxed{y = \sin 2t}$

23. $\frac{40.5}{s(s^2-9)} = \frac{40.5}{s(s^2-3^2)} = \frac{13.5}{s} \cdot \frac{3}{s^2-3^2}$

$\mathcal{L}^{-1}\left(\frac{13.5}{s}\right) * \mathcal{L}^{-1}\left(\frac{3}{s^2-3^2}\right) = 13.5 * \sinh(3t)$

$\int_0^t 13.5 \cdot \sinh(3\tau)\, d\tau = \frac{13.5}{3}\cosh(3\tau)\Big|_0^t$

$4.5\cosh(3t) - 4.5\cosh(0)$

$= \boxed{4.5\left(\cosh(3t) - 1\right)}$