Name: Zhu Shi
PUID: 0025900006

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Homework 7

P237. 3. Solution:

The subsidiary equation is as follows:

$s^2Y - sy(0) - y'(0) + 4Y = e^{-\pi s}$ thus $s^2Y - 8s + 4Y = e^{-\pi s}$

Therefore,

$$Y = \frac{e^{-\pi s} + 8s}{s^2+4} = \frac{1}{s^2+4}e^{-\pi s} + 8\cdot\frac{s}{s^2+4}$$

From Table 6.1, we see that:

$$L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin 2t \qquad L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t$$

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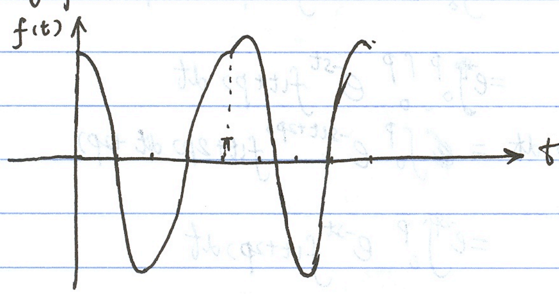
Hence: $f(t) = L^{-1}(Y) = \frac{1}{2}\sin[2(t-\pi)]\,u(t-\pi) + 8\cos 2t$

$= \frac{1}{2}\sin 2t \cdot u(t-\pi) + 8\cos 2t$

$$= \begin{cases} 8\cos 2t & 0<t<\pi \\ \frac{1}{2}\sin 2t + 8\cos 2t & t>\pi \end{cases}$$

The graph is shown below:



10. Solution:

The subsidiary equation is as follows:

$$[s^2Y - sy(0) - y'(0)] + 5[sY - f(0)] + 6Y = e^{-\frac{\pi}{2}s} - \frac{s}{s^2+1}e^{-\pi s}$$

thus,

$$(s^2+5s+6)Y = e^{-\frac{\pi}{2}s} - \frac{s}{s^2+1}e^{-\pi s}$$

$$\Rightarrow Y = \frac{1}{s^2+5s+6}e^{-\frac{\pi}{2}s} - \frac{s}{(s^2+5s+6)(s^2+1)}e^{-\pi s}$$

$$= \frac{1}{s+2}e^{-\frac{\pi}{2}s} - \frac{1}{s+3}e^{-\frac{\pi}{2}s} + \frac{2}{5(s+2)}e^{\pi s} - \frac{3}{10(s+3)}e^{-\pi s} - \frac{1}{10}\frac{s}{s^2+1}e^{-\pi s} - \frac{1}{10}\frac{1}{s^2+1}e^{-\pi s}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}(Y)$$

$$= \left[e^{-2(t-\frac{\pi}{2})} - e^{-3(t-\frac{\pi}{2})}\right]u(t-\tfrac{\pi}{2}) + \left[\tfrac{2}{5}e^{-2(t-\pi)} - \tfrac{3}{10}e^{-3(t-\pi)} - \tfrac{1}{10}\cos(t-\pi) - \tfrac{1}{10}\sin(t-\pi)\right]u(t-\pi)$$

$$= \left[e^{-2(t-\frac{\pi}{2})} - e^{-3(t-\frac{\pi}{2})}\right]u(t-\tfrac{\pi}{2}) + \left[\tfrac{2}{5}e^{-2(t-\pi)} - \tfrac{3}{10}e^{-3(t-\pi)} + \tfrac{1}{10}\cos t + \tfrac{1}{10}\sin t\right]u(t-\pi)$$

Below is the graph of $y(t)$:

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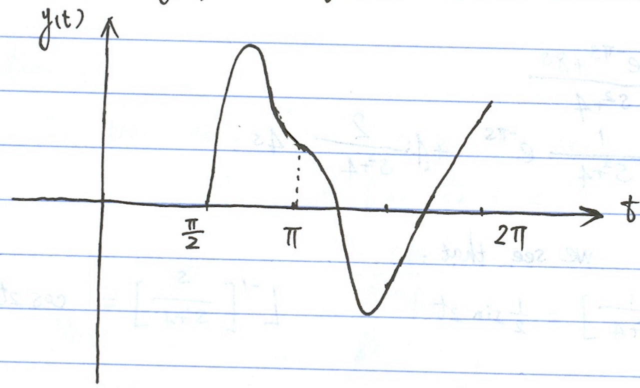

14(a). Proof:

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^p e^{-st} f(t)\,dt + \int_p^{2p} e^{-st} f(t)\,dt + \cdots + \int_{np}^{np+p} e^{-st} f(t)\,dt + \cdots$$

where:

$$\int_p^{2p} e^{-st} f(t)\,dt = \int_0^p e^{-s(t+p)} f(t+p)\,d(t+p)$$

$$= e^{-sp}\int_0^p e^{-st} f(t+p)\,dt$$

$$\int_{2p}^{3p} e^{-st} f(t)\,dt = \int_0^p e^{-s(t+2p)} f(t+2p)\,d(t+2p)$$

$$= e^{-2sp}\int_0^p e^{-st} f(t+2p)\,dt$$

...

...

$$\int_{np}^{(n+1)p} e^{-st} f(t)\,dt = e^{-nsp}\int_0^p e^{-st} f(t+np)\,dt$$

...

Since $f(t)$ is a continuous periodic function with period p,

$$f(t) = f(t+p) = f(t+2p) = \cdots = f(t+np) = \cdots$$

Thus: $\mathcal{L}(f) = \int_0^p e^{-st} f(t)\,dt + e^{-sp}\int_0^p e^{-st} f(t)\,dt + \cdots + e^{-nsp}\int_0^p e^{-st} f(t)\,dt + \cdots$

$$= (1 + e^{-sp} + e^{-2sp} + \cdots + e^{-nsp} + \cdots)\int_0^p e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)\, dt$$

This completes the proof.

14(b) Proof:

From (1), we know that for a periodic function with period p,

$$L(f) = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

Therefore, in this problem,

$$L(f) = \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\, dt.$$

Let $F(s) = \int_0^{\pi/\omega} e^{-st} \sin \omega t\, dt$, then:

$$F(s) = -\frac{1}{\omega} \int_0^{\frac{\pi}{\omega}} e^{-st}\, d(\cos \omega t)$$

$$= -\frac{1}{\omega}\left[\cos \omega t\, e^{-st} \Big|_0^{\frac{\pi}{\omega}} - \int_0^{\frac{\pi}{\omega}} \cos \omega t\, d\left(e^{-st}\right) \right]$$

$$= -\frac{1}{\omega}\left[-e^{-s\pi/\omega} - 1 + s\int_0^{\pi/\omega} e^{-st} \cos \omega t\, dt \right]$$

$$= \frac{1}{\omega}\left(e^{-\pi s/\omega} + 1\right) - \frac{s}{\omega^2} \int_0^{\pi/\omega} e^{-st}\, d(\sin \omega t)]$$

$$= \frac{1}{\omega}\left(e^{-s\pi/\omega} + 1\right) - \frac{s}{\omega^2}\left[e^{-st} \sin \omega t \Big|_0^{\pi/\omega} - \int_0^{\pi/\omega} \sin \omega t\, d(e^{-st}) \right]$$

$$= \frac{1}{\omega}\left(e^{-s\pi/\omega} + 1\right) - \frac{s^2}{\omega^2} F(s)$$

$$\Rightarrow F(s) = \frac{\omega}{\omega^2 + s^2}\left(e^{-s\pi/\omega} + 1\right)$$

Hence, $L(f) = \dfrac{1}{1-e^{-2\pi s/\omega}} \cdot \dfrac{\omega}{s^2+\omega^2}\left(1 + e^{-s\pi/\omega}\right)$

$$= \frac{1}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$$

This completes the proof.

P237. 7. Solution:

$$t * e^t = \int_0^t \tau e^{(t-\tau)}\, d\tau$$

$$= e^t \int_0^t \tau e^{-\tau}\, d\tau$$

$$= -e^t \int_0^t \tau \, d(e^{-\tau})$$
$$= -e^t \left[\tau e^{-\tau} \Big|_0^t - \int_0^t e^{-\tau} d\tau \right]$$
$$= -e^t (t e^{-t} + e^{-t} - 1)$$
$$= e^t - t - 1$$

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8. Solution:

With the convolution theorem, we can obtain:

$$Y(s) + 4 Y(s) \cdot \frac{1}{s^2} = 2 \cdot \frac{1}{s^2}$$

$$\Rightarrow Y(s) = \frac{2}{s^2+4}$$

Since $L^{-1}[Y(s)] = \sin 2t$,

the solution is:

$$y(t) = \sin 2t$$

23. Solution:

$$L^{-1}\left[\frac{40.5}{s(s^2-9)} \right] = L^{-1}\left[13.5 \cdot \frac{1}{s} \cdot \frac{3}{s^2-9} \right]$$

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$$= 13.5 \cdot 1 * \sinh 3t$$
$$= 13.5 \int_0^t \sinh 3\tau \cdot 1 \, d\tau$$
$$= 13.5 \int_0^t \frac{e^{3\tau} - e^{-3\tau}}{2} d\tau$$
$$= 4.5 \cdot \frac{e^{3\tau} + e^{-3\tau}}{2} \Big|_0^t$$
$$= 4.5(\cosh 3t - 1)$$