Jose Quinones-Torres DUE: 10/24/2012

Homework # 8

Sec. 6.6 p 241

Showing the details of your work, find $\mathcal{L}(f)$ if $f(t)$ equals:

(3.) $f(t) = \frac{1}{2} t e^{-3t}$, $\mathcal{L}(e^{-3t}) = \frac{1}{s+3}$

$$\mathcal{L}\left(\frac{1}{2} t e^{-3t}\right) = \frac{1}{2}\left(-\frac{d}{ds}\frac{1}{s+3}\right) = \frac{-1}{2}\frac{d}{ds}(s+3)^{-1} = \frac{-1}{2}(-1)(s+3)^{-2}$$

$$\boxed{\mathcal{L}\left(\tfrac{1}{2} t e^{-3t}\right) = {}^{1/2}/_{(s+3)^2}}$$

(8.) $f(t) = te^{-kt}\sin t$, $\mathcal{L}(e^{-kt}\sin t) = \frac{1}{(s+k)^2+1}$

$$\mathcal{L}(te^{-kt}\sin t) = -\frac{d}{ds}\frac{1}{(s+k)^2+1} = -\frac{d}{ds}\left[(s+k)^2+1\right]^{-1} = -(-1)(2)(s+k)\left[(s+k)^2+1\right]^{-2}$$

$$\boxed{\mathcal{L}(te^{-kt}\sin t) = 2(s+k)/((s+k)^2+1)^2}$$

(10.) $f(t) = t^n e^{kt}$, $\mathcal{L}(t^n) = n!/s^{n+1}$, $\mathcal{L}(e^{kt}f(t)) = F(s-k)$

$$\therefore \boxed{\mathcal{L}(t^n e^{kt}) = n!/(s-k)^{n+1}}$$

Using differentiation, integration, s-shifting, or convolution, and showing the details, find $f(t)$ if $\mathcal{L}(f)$ equals:

16. $F(s) = 2s+6/(s^2+6s+10)^2 = 2(s+3)/[(s+3)^2+1]^2$

$\mathcal{L}(f) = 2(s+3)/[(s+3)^2+1]^2$

$$f(t) = \mathcal{L}^{-1}\left(2(s+3)/[(s+3)^2+1]^2\right) = e^{-3t}\mathcal{L}^{-1}\left(2s/(s^2+1)^2\right)$$
$$= e^{-3t}\mathcal{L}^{-1}\left(-\frac{d}{ds}\frac{1}{(s^2+1)}\right) = e^{-3t}t\,\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$$

$$\boxed{f(t) = te^{-3t}\sin t}$$

Sec. 6.7 p 246

Using the Laplace transform and showing the details of your work, solve the IVP:

3. $y_1' = -y_1 + 4y_2 \quad y_1(0) = 3$
$y_2' = 3y_1 - 2y_2 \quad y_2(0) = 4$

$$\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Y(s) = (sI - A)^{-1}\bar{y}(0)$$
$$= \begin{bmatrix} s+1 & -4 \\ -3 & s+2 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{(s+1)(s+2)-12}\begin{bmatrix} s+2 & 4 \\ 3 & s+1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \frac{1}{s^2+3s-10}\begin{bmatrix} 3s+22 \\ 4s+13 \end{bmatrix} = \frac{1}{(s+5)(s-2)}\begin{bmatrix} 3s+22 \\ 4s+13 \end{bmatrix}$$

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$$Y_1(s) = \frac{3s+22}{(s+5)(s-2)} = \frac{A}{s+5} + \frac{B}{s-2} = \frac{4}{s-2} - \frac{1}{s+5}$$

$$3s+22 = A(s-2) + B(s+5)$$

$$s=2: \ 28 = 7B \Rightarrow B = 4$$

$$s=-5: \ 7 = -7A \Rightarrow A = -1$$

$$Y_2(s) = \frac{4s+13}{(s+5)(s-2)} = \frac{A}{s+5} + \frac{B}{s-2} = \frac{1}{s+5} + \frac{3}{s-2}$$

$$4s+13 = A(s-2) + B(s+5)$$

$$s=2: \ 21 = 7B \Rightarrow B = 3$$

$$s=-5: \ -7 = -7A \Rightarrow A = 1$$

$$\boxed{\begin{aligned} y_1(t) &= 4e^{2t} - e^{-5t} \\ y_2(t) &= 3e^{2t} + e^{-5t} \end{aligned}}$$

(12) $y_1'' = -2y_1 + 2y_2 \quad y_1(0) = 1 \quad y_1'(0) = 0$

$y_2'' = 2y_1 - 5y_2 \quad y_2(0) = 3 \quad y_2'(0) = 0$

$$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Y(s) = (s^2 I - A)^{2} \left(s\bar{y}(0) + \bar{y}'(0) \right)$$

$$= \begin{bmatrix} s^2+2 & -2 \\ -2 & s^2+5 \end{bmatrix}^{-1} \begin{bmatrix} s \\ 3s \end{bmatrix} = \frac{1}{(s^2+2)(s^2+5)-4} \begin{bmatrix} s^2+5 & 2 \\ 2 & s^2+2 \end{bmatrix} \begin{bmatrix} s \\ 3s \end{bmatrix}$$

$$= \frac{1}{s^4+7s^2+6} \begin{bmatrix} s^3+11s \\ 3s^3+8s \end{bmatrix} = \frac{s}{(s^2+1)(s^2+6)} \begin{bmatrix} s^2+11 \\ 3s^2+8 \end{bmatrix}$$

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$$Y_1(s) = \frac{s(s^2+11)}{(s^2+1)(s^2+6)} = \frac{As+B}{s^2+6} + \frac{Cs+D}{s^2+1} = \frac{2s}{s^2+1} - \frac{s}{s^2+6}$$

$$s(s^2+11) = (As+B)(s^2+1) + (Cs+D)(s^2+6)$$

$$s^3 + 11s = (A+C)s^3 + (B+D)s^2 + (A+6C)s + (B+6D)$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 6 & 0 & 11 \\ 0 & 1 & 0 & 6 & 0 \end{array}\right] \qquad \begin{array}{l} A = -1 \\ B = 0 \\ C = 2 \\ D = 0 \end{array}$$

$$Y_2(s) = \frac{s(3s^2+8)}{(s^2+1)(s^2+6)} = \frac{As+B}{s^2+6} + \frac{Cs+D}{s^2+1} = \frac{2s}{s^2+6} + \frac{s}{s^2+1}$$

$$s(3s^2+8) = (As+B)(s^2+1) + (Cs+D)(s^2+6)$$

$$3s^3 + 8s = (A+C)s^3 + (B+D)s^2 + (A+6C)s + (B+6D)$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 6 & 0 & 8 \\ 0 & 1 & 0 & 6 & 0 \end{array}\right] \qquad \begin{array}{l} A = 2 \\ B = 0 \\ C = 1 \\ D = 0 \end{array}$$

$$\boxed{\begin{array}{l} y_1(t) = 2\cos t - \cos\sqrt{6}\,t \\ y_2(t) = \cos t + 2\cos\sqrt{6}\,t \end{array}}$$

Sec. 12.12 p602

Solve by Laplace Transforms

(5) $x\frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial t} = xt \qquad \omega(x,0) = 0 \text{ if } x \geq 0$

$\omega(0,t) = 0 \text{ if } t \geq 0$

$$x\frac{\partial}{\partial x}W(x,s) + [sW(x,s) - \omega(x,0)] = \frac{x}{s^2}$$

$$x\frac{\partial}{\partial x}W(x,s) + sW(x,s) = \frac{x}{s^2}$$

$$xW' + sW = x/s^2 \Rightarrow W' + \frac{s}{x}W = \frac{1}{s^2} \qquad p = s/x,\ q = 1/s^2$$

$$W' + pW = q$$

$$W(x,s) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} q\, dx + c\right]$$

$$\int s/x\, dx = s\ln x + c \qquad e^{-s\ln x - c} = e^{-c}/x^s$$

$$\int e^{c} x^{s} \cdot \frac{1}{s^2} dx = e^{c}/s^2 \cdot x^{s+1}/(s+1)$$

$$W(x,s) = 1/e^{c}x^{s} \cdot e^{c}/s^2 \cdot x^{s+1}/(s+1) = x/s^2(s+1)$$

$$\frac{x}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{c}{s+1} \qquad x = As(s+1) + B(s+1) + cs^2$$

$s = 0: \ B = x$

$$= \frac{x}{s+1} - \frac{x}{s} + \frac{x}{s^2} \qquad s = -1: \ c = x$$

$s = 1: \ 2A + 2B + x = x \Rightarrow A = -x$

$$\omega(x,t) = xe^{-t} + xt - x \qquad \boxed{\omega(x,t) = x(e^{-t} + t - 1)}$$

Sec. 11.1 p482

Find the Fourier series of the given function f(x), which is assumed to have period 2π. Show the details of your work. Sketch or graph the partial sum up to that including $\cos 5x$ and $\sin 5x$.

(12) $f(x) = |x|$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_0^{\pi} x\,dx\right] = \frac{1}{2\pi}\left[\frac{-x^2}{2}\Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{+\pi^2}{2} + \frac{\pi^2}{2}\right] = \pi/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} x\cos nx\,dx\right]$$

$u = x \qquad dv = \cos nx$

$du = dx \qquad v = \frac{1}{n}\sin nx$

$$a_n = \frac{2}{\pi}\left[\frac{x}{n}\sin nx\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin nx\,dx\right] = \frac{2}{\pi}\left[-\frac{1}{n}\cdot -\frac{1}{n}\cos nx\Big|_0^{\pi}\right]$$

$$= \frac{2}{\pi n^2}\left[\cos n\pi - \cos 0\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0, & n = 2, 4, 6, \dots \\ -4/\pi n^2, & n = 1, 3, 5, \dots \end{cases}$$

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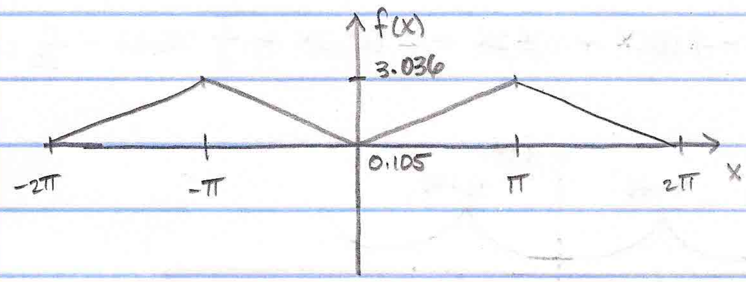
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = 0 \quad \text{by symmetry}$$

$$\boxed{|x| \simeq \frac{\pi}{2} + \sum_{n=0}^{\infty} {}^{-4}\!/_{\pi(2n+1)^2}\cos(2n+1)x}$$

$$|x| \simeq \frac{\pi}{2} + \frac{-4}{\pi(1)^2}\cos x + \frac{-4}{\pi(3)^2}\cos 3x + \frac{-4}{\pi(5)^2}\cos 5x$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \frac{4}{25\pi}\cos 5x$$



⑭ $f(x) = x^2 \quad (-\pi < x < \pi)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] = \frac{1}{2\pi}\cdot\frac{2\pi^3}{3} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx \qquad u = x^2 \qquad dv = \cos nx\,dx$$

$$du = 2x\,dx \qquad v = \frac{1}{n}\sin nx$$

$$a_n = \frac{1}{\pi}\left[\frac{x^2}{n}\sin nx\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{2x}{n}\sin nx\,dx\right] \qquad u = \frac{2}{n}x \qquad dv = \sin nx\,dx$$

$$du = dx \qquad v = -\frac{1}{n}\cos nx$$

$$a_n = \frac{1}{\pi}\left[\frac{2x}{n^2}\cos nx\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{1}{n}\cos nx\,dx\right]$$

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$$a_n = \frac{1}{\pi}\left[\frac{2x}{n^2}\cos nx \Big|_{-\pi}^{\pi} - \frac{1}{n^2}\sin nx \Big|_{-\pi}^{\pi}\right]$$

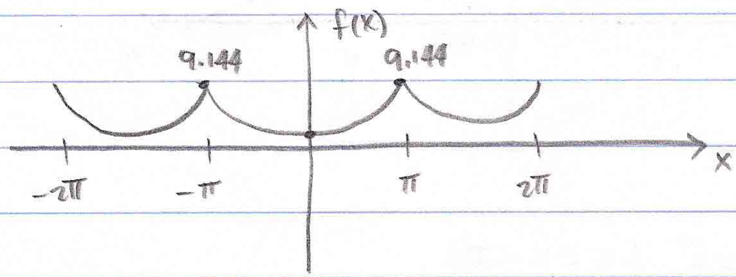
$$= \frac{1}{\pi}\left[\frac{2\pi}{n^2}\cos n\pi + \frac{2\pi}{n^2}\cos -n\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\cos n\pi\right] = \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

$$b_n = \int_{-\pi}^{\pi} x^2 \sin nx \, dx = 0 \quad \text{by symmetry}$$

$$x^2 \approx \pi^2/3 + \sum_{n=1}^{\infty} 4/n^2 \, (-1)^n \cos nx$$

$$x^2 \approx \pi^2/3 - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \frac{1}{4}\cos 4x - \frac{4}{25}\cos 5x$$



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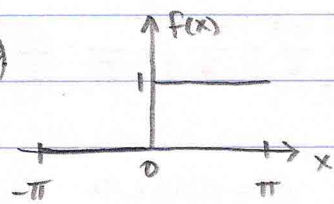

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 0 \, dx + \frac{1}{2\pi}\int_{0}^{\pi} 1 \, dx$$

$$= \frac{1}{2\pi} x \Big|_0^{\pi} = \frac{1}{2\pi}\pi = \frac{1}{2}$$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} \cos nx\,dx$$

$$= \frac{1}{n\pi}\sin nx \Big|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} \sin nx\,dx$$

$$= \frac{-1}{n\pi}\cos nx \Big|_0^{\pi} = \frac{-1}{n\pi}\left[\cos n\pi - 1\right] = \frac{1}{n\pi}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 2/n\pi, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}$$

$$\boxed{f(x) \cong \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{\pi(2n+1)} \sin (2n+1)x}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x$$