Name: Zhu Shi

PUID: 00259-00006

40/40

Homework 8

P241. 3.

Solution:

$$L[\tfrac{1}{2}e^{-3t}] = \tfrac{1}{2}L[e^{-3t}] = \tfrac{1}{2}\cdot\frac{1}{s+3} = \frac{1}{2(s+3)}$$

$$\Rightarrow L[\tfrac{1}{2}te^{-3t}] = L[t\cdot\tfrac{1}{2}e^{-3t}] = -\left[\frac{1}{2(s+3)}\right]' = \frac{1}{2(s+3)^2}$$

8.

Solution:

$$L[e^{-kt}\sin t] = \frac{1}{(s+k)^2+1}$$

$$\Rightarrow L[t\cdot e^{-kt}\sin t] = -\left[\frac{1}{(s+k)^2+1}\right]' = \frac{2(s+k)}{[(s+k)^2+1]^2}$$

10. Solution:

$$L[t^n e^{kt}] = -\{L[t^{n-1}e^{kt}]\}' = (-1)^2\{L[t^{n-2}e^{kt}]\}'' = \cdots = (-1)^n\{L[e^{kt}]\}^{(n)}$$

~~where~~ Since $L[e^{kt}] = \frac{1}{s-k}$,

$$\{L[e^{kt}]\}' = -\frac{1}{(s-k)^2}$$

$$\{L[e^{kt}]\}'' = (-1)\cdot(-2)\cdot\frac{1}{(s-k)^3}$$

...

$$\{L[e^{kt}]\}^{(n)} = (-1)^n\cdot\frac{n!}{(s-k)^{n+1}}$$

$$\Rightarrow L[t^n e^{kt}] = (-1)^n\cdot(-1)^n\cdot\frac{n!}{(s-k)^{n+1}} = \frac{n!}{(s-k)^{n+1}}$$

16. Solution:

$$\frac{2s+6}{(s^2+6s+10)^2} = \frac{2(s+3)}{[(s+3)^2+1]^2}$$

$$L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

Since $\left(\frac{1}{s^2+1}\right)' = -\frac{2s}{(s^2+1)^2}$, we have:

$$L^{-1}\left[-\frac{2s}{(s^2+1)^2}\right] = -t\sin t$$

which means:

$$L^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = t\sin t$$

Using s-shifting theorem, we have:

$$L^{-1}\left[\frac{2s+6}{(s^2+6s+10)^2}\right] = L^{-1}\left\{\frac{2(s+3)}{[(s+3)^2+1]^2}\right\}$$

$$= e^{-3t}t\sin t$$

10/10

p246. 3. Solution:

With inverse Laplace transform we have:

$$\begin{cases} sY_1 - y_1(0) = -Y_1 + 4Y_2 \\ sY_2 - y_2(0) = 3Y_1 - 2Y_2 \end{cases}, \text{ where } y_1(0)=3,\ y_2(0)=4.$$

$$\Rightarrow \begin{cases} sY_1 - 3 = -Y_1 + 4Y_2 \\ sY_2 - 4 = 3Y_1 - 2Y_2 \end{cases}$$

$$\Rightarrow \begin{cases} Y_1 = \dfrac{3s+22}{s^2+3s-10} = \dfrac{3(s+\frac{3}{2}) + \frac{35}{2}}{(s+\frac{3}{2})^2 - (\frac{7}{2})^2} \\ Y_2 = \dfrac{4s+13}{s^2+3s-10} = \dfrac{4(s+\frac{3}{2})+7}{(s+\frac{3}{2})^2-(\frac{7}{2})^2} \end{cases}$$

$$\Rightarrow y_1(t) = L^{-1}(Y_1) = 3\cdot\frac{e^{\frac{7}{2}t}-e^{-\frac{7}{2}t}}{2}\cdot e^{-\frac{3}{2}t} + 5\cdot\frac{e^{\frac{7}{2}t}-e^{-\frac{7}{2}t}}{2}e^{-\frac{3}{2}t} = 4e^{2t} - e^{-5t}$$

$$y_2(t) = L^{-1}(Y_2) = 4\cosh(\tfrac{7}{2}t)\cdot e^{-\frac{3}{2}t} + 2\sinh(\tfrac{7}{2}t)\cdot e^{-\frac{3}{2}t}$$

$$= 3e^{2t} + e^{-5t}$$

Therefore, the solution is:

$$\begin{cases} y_1(t) = 4e^{2t} - e^{-5t} \\ y_2(t) = 3e^{2t} + e^{-5t} \end{cases}$$

12. Solution:

With the inverse Laplace transform we have:

$$\begin{cases} s^2Y_1 - sy_1(0) - y_1'(0) = -2Y_1 + 2Y_2 \\ s^2Y_2 - sy_2(0) - y_2'(0) = 2Y_1 - 5Y_2 \end{cases} \text{ where } y_1(0)=1,\ y_1'(0)=0,\ y_2(0)=3,\ y_2'(0)=0.$$

$$\Rightarrow \begin{cases} s^2Y_1 - s = -2Y_1 + 2Y_2 \\ s^2Y_2 - 3s = 2Y_1 - 5Y_2 \end{cases}$$

$$\Rightarrow \begin{cases} Y_1 = \frac{s(s^2+11)}{(s^2+6)(s^2+1)} = -\frac{s}{s^2+6} + 2\frac{s}{s^2+1} \\ Y_2 = \frac{s(s^2+8)}{(s^2+6)(s^2+1)} = \frac{s}{s^2+1} + \frac{2s}{s^2+6} \end{cases}$$

$$\Rightarrow y_1(t) = L^{-1}[Y_1] = -\cos\sqrt{6}t + 2\cos t$$

$$y_2(t) = L^{-1}(Y_2) = \cos t + 2\cos\sqrt{6}t$$

10/10

P602. 5. Solution:

$$L(x\frac{\partial w}{\partial x}) = x\frac{\partial}{\partial x}[L(w)]$$
$$= x\frac{\partial}{\partial x}W$$
$$L(\frac{\partial w}{\partial t}) = sW(x,s) - w(x,0)$$
$$= sW$$
$$L(xt) = x\frac{1}{s^2}$$

Therefore, from $x\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} = xt$ we can get:

10/10

$$xW' + sW = \frac{x}{s^2}$$
$$\Rightarrow W' + \frac{s}{x}W = \frac{1}{s^2}$$
$$\Rightarrow W(x,s) = e^{-\int\frac{s}{x}dx}\left[\int\frac{x}{s^2}e^{\int\frac{s}{x}dx}dx + C(s)\right]$$
$$= x^{-s}\left[\frac{1}{s^2(s+1)}x^{s+1} + C(s)\right]$$
$$= \frac{1}{s^2(s+1)}x + C(s)x^{-s}$$

Since $w(0,t)=0$, we can get:

$$W(0,s) = 0$$

Therefore $C(s)=0$

Thus: $W(x,s) = \frac{1}{s^2(s+1)}x$

$$\Rightarrow w(x,t) = L^{-1}\left[\frac{1}{s^2(s+1)}x\right]$$
$$= L^{-1}\left[\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right)x\right]$$
$$= (t-1+e^{-t})x$$

P482. 12. Solution:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$
$$= \frac{2}{n^2\pi}(\cos n\pi - 1) = \begin{cases} 0, & \text{if } n \text{ is an even number.} \\ -\frac{4}{n^2\pi}, & \text{if } n \text{ is an odd number.} \end{cases}$$

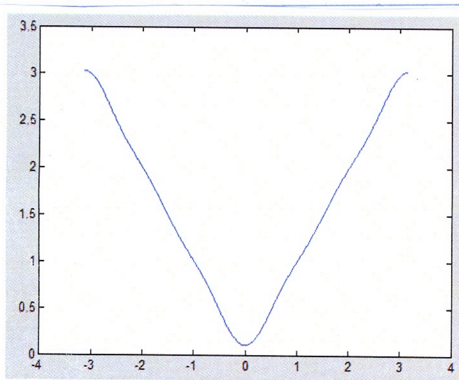
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$
$$= 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$
$$= \frac{1}{2}\pi$$

Therefore, the Fourier series is:

$$\frac{1}{2}\pi + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi}(\cos n\pi - 1)\cos nx\right]$$
$$= \frac{1}{2}\pi - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \frac{4}{25\pi}\cos 5x + \cdots + \frac{2}{n^2\pi}(\cos n\pi - 1)\cos nx + \cdots$$

The graph ~~up~~ ~~to~~ of the partial sums up to that including $\cos 5x$ and $\sin 5x$ is as follows:



14. Solution:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$
$$= \frac{1}{\pi}\left[\frac{1}{n}\int_{-\pi}^{\pi} x^2\,d(\sin nx)\right]$$
$$= \frac{1}{n\pi}\left[x^2\sin nx\Big|_{-\pi}^{\pi} - 2\int_{-\pi}^{\pi} x\sin nx\,dx\right]$$
$$= \frac{1}{n\pi}\left[2\pi^2\sin n\pi\right] + \frac{2}{n^2\pi}\int_{-\pi}^{\pi} x\,d(\cos nx)$$
$$= \frac{2\pi}{n}\underbrace{\sin n\pi}_{0} + \frac{2}{n^2\pi}\left[x\cos nx\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\cos nx\,dx\right]$$
$$= \frac{4}{n^2}\cos n\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$
$$= 0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

Therefore, the Fourier series is:

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos n\pi\cdot\cos nx\right]$$
$$= \frac{\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \frac{1}{4}\cos 4x - \frac{4}{25}\cos 5x + \cdots$$