Jose Quinones-Torres DUE: 10/31/2012

Homework #9

Sec. 11.2 p 490

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.

(11.) $f(x) = x^2, \quad (-1 < x < 1), \quad p = 2 \quad \Rightarrow L = 1$

$f(-x) = (-x)^2 = x^2 \quad \therefore$ even function $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{2(1)} \int_{-1}^{1} f(x)\,dx = \frac{1}{2(1)} \int_{-1}^{1} x^2\,dx = \frac{1}{2(1)} \cdot \frac{x^3}{3} \Big|_{-1}^{1} = \frac{1}{2}\left[\frac{1}{3} + \frac{1}{3}\right] = \frac{1}{3}$$

$$a_n = \frac{1}{1} \int_{-1}^{1} f(x) \cos n\pi x\,dx = \int_{-1}^{1} x^2 \cos n\pi x\,dx \qquad u = x^2 \qquad dv = \cos n\pi x\,dx$$

$$du = 2x\,dx \qquad v = \frac{1}{n\pi} \sin n\pi x$$

$$a_n = \frac{1}{n\pi} x^2 \sin n\pi x \Big|_{-1}^{1} - \int_{-1}^{1} \frac{1}{n\pi} 2x \sin n\pi x\,dx \qquad u = x \qquad dv = \sin n\pi x\,dx$$

$$du = dx \qquad v = \frac{-1}{n\pi} \cos n\pi x$$

$$a_n = \frac{2}{n^2\pi^2} x \cos n\pi x \Big|_{-1}^{1} + \frac{2}{n\pi} \int_{-1}^{1} \frac{-1}{n\pi} \cos n\pi x\,dx$$

$$= \frac{2}{n^2\pi^2}\left[\cos n\pi + \cos -n\pi\right] - \frac{2}{n^2\pi^2} \cdot \frac{1}{n\pi} \cos n\pi x \Big|_{-1}^{1}$$

$$= \frac{4}{n^2\pi^2} \cos n\pi - \frac{2}{n^3\pi^3}\left[\cos n\pi - \cos -n\pi\right] = \frac{4}{n^2\pi^2} \cos n\pi = \frac{4}{n^2\pi^2} (-1)^n$$

$$\boxed{f(x) \cong \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n \cos n\pi x}$$

(20) Numeric Values. Using Prob. 11, show that $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots = \frac{1}{6}\pi^2$

From Prob. 11 result: $\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n \cos n\pi x$

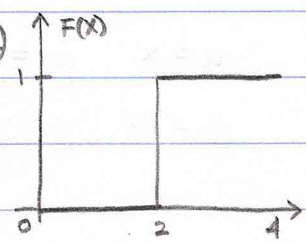
For $x=1$, $f(x)=1$, therefore

$$\frac{1}{3}+\frac{4}{\pi^2}\left[-\cos\pi+\frac{1}{4}\cos 2\pi-\frac{1}{9}\cos 3\pi+\frac{1}{16}\cos 4\pi+\dots\right]=1$$
$$\frac{1}{3}+\frac{4}{\pi^2}\left[1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots\right]=1$$

$$\frac{4}{\pi^2}\left[1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots\right]=\frac{2}{3} \Rightarrow 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots=\frac{1}{6}\pi^2 \checkmark$$

Find (a) the Fourier cosine series, (b) the Fourier sine series. Sketch f(x) and its two periodic extensions. Show the details.

(24)




$$F(X)=\begin{cases}0, & 0<X<2\\ 1, & 2<X<4\end{cases}, \quad L=4$$

$$a_0=\frac{1}{4}\int_0^4 F(x)\,dx=\frac{1}{4}\int_2^4 1\,dx=\frac{1}{4}x\Big|_2^4=\frac{1}{4}[4-2]=\frac{1}{2}$$

(a) $$a_n=\frac{2}{4}\int_0^4 F(x)\,dx=\frac{2}{4}\int_2^4 \cos\frac{n\pi}{4}x\,dx=\frac{2}{n\pi}\sin\frac{n\pi}{4}x\Big|_2^4$$

$$=\frac{2}{n\pi}\left[\sin n\pi-\sin n\frac{\pi}{2}\right]=\frac{2}{n\pi}\left[-\sin n\frac{\pi}{2}\right]=\frac{-2}{(2n+1)\pi}(-1)^n$$

$$F.S.=\frac{1}{2}-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}(-1)^n\cos((2n+1)x)$$

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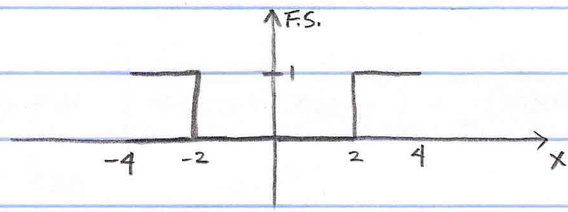
b) $b_n = \frac{2}{4}\int_0^4 F(x)\sin\frac{n\pi}{4}x\,dx = \frac{2}{4}\int_2^4 \sin\frac{n\pi}{4}x\,dx = \frac{-2}{n\pi}\cos\frac{n\pi}{4}x\Big|_2^4$

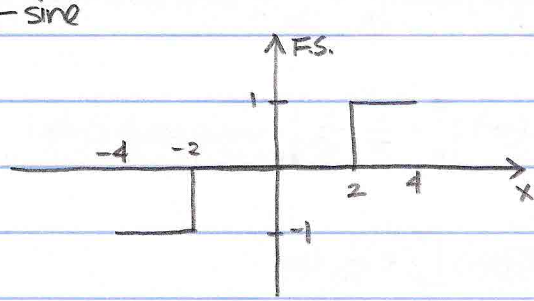
$= \frac{-2}{n\pi}\left[\cos n\pi - \cos\frac{n\pi}{2}\right] = \frac{-2}{n\pi}\left[(-1)^n - \cos\frac{n\pi}{2}\right]$

$F.S. = \frac{-2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\sin n\pi x$

F - cosine




F - sine



(29) $f(x) = \sin x$ , $(0 < x < \pi)$ , $L = \pi$

(a) $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = -\frac{1}{\pi}\cos x\,\Big|_0^{\pi}$

$= -\frac{1}{\pi}[-1-1] = \frac{2}{\pi}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos\frac{n\pi}{\pi}x\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$ $\quad u = \sin x \quad dv = \cos(nx)dx$

$du = \cos x\,dx \quad v = +\frac{1}{n}\sin(nx)$

$a_n = \frac{2}{\pi}\left[+\frac{1}{n}\cancel{\sin x \sin(nx)}^{\,0}\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\cos x \sin(nx)\,dx\right]$ $\quad u = \cos x \quad dv = \sin(nx)dx$

$du = -\sin x\,dx \quad v = -\frac{1}{n}\cos(nx)$

$a_n = \frac{2}{\pi}\left[-\frac{1}{n}\left(-\frac{1}{n}\cos(nx)\cos x\,\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin x\cos(nx)\,dx\right)\right]$

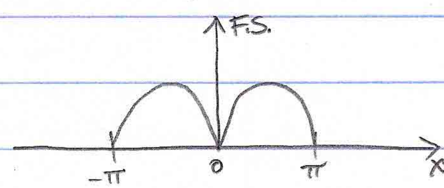
$= \frac{2}{\pi}\cdot\frac{1}{n^2}\cos x\cos(nx)\,\Big|_0^{\pi} + \frac{2}{\pi}\cdot\frac{1}{n^2}\int_0^{\pi}\sin x\cos(nx)\,dx$

$= \frac{2}{\pi}\cdot\frac{1}{n^2}\cos x\cos(nx)\,\Big|_0^{\pi} + \frac{1}{n^2}a_n$

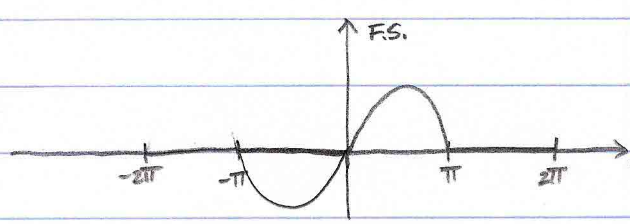
$a_n - \frac{1}{n^2}a_n = \frac{2}{\pi}\cdot\frac{1}{n^2}\cos x\cos(nx)\,\Big|_0^{\pi} \Rightarrow a_n = \frac{2}{\pi(n^2-1)}\cos x\cos(nx)\,\Big|_0^{\pi}$

$a_n = \frac{2}{\pi(n^2-1)}\left[-(-1)^n - 1\right] = \frac{-2}{\pi(n^2-1)}\left[1 + (-1)^n\right]$

$F.S. = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{1}{n^2-1}\left[1+(-1)^n\right]\cos(n\pi x)$




(b) $F.S. = \sin x$



Sec. 11.3 p 494

(3) Change of spring and damping. In example 1, what happens to the amplitudes $C_n$ if we take a stiffer spring, say, of $k = 49$? If we increase the damping?

$$y'' + cy' + ky = r(t)$$

$$r(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt)\right)$$
$$y(t) = c_0 + \sum_{n=1}^{\infty} \left(c_n \cos(nt) + d_n \sin(nt)\right)$$
$$y'(t) = \sum_{n=1}^{\infty} \left(-nc_n \sin(nt) + nd_n \cos(nt)\right)$$
$$y''(t) = \sum_{n=1}^{\infty} \left(-n^2 c_n \cos(nt) - n^2 d_n \sin(nt)\right)$$

$$\sum_{n=1}^{\infty} \left(-n^2 c_n \cos(nt) - n^2 d_n \sin(nt)\right) + c \sum_{n=1}^{\infty} \left(nd_n \cos(nt) - nc_n \sin(nt)\right) +$$
$$kc_0 + k \sum_{n=1}^{\infty} \left(c_n \cos(nt) + d_n \sin(nt)\right) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt)\right)$$

$$kc_0 = a_0 \Rightarrow c_0 = a_0/k$$

$$-n^2 c_n + cnd_n + kc_n = a_n \qquad (k - n^2) c_n + cnd_n = a_n$$
$$-n^2 d_n - cnc_n + kd_n = b_n \qquad -cnc_n + (k - n^2) d_n = b_n$$

$$\begin{bmatrix} k-n^2 & cn \\ -cn & k-n^2 \end{bmatrix} \begin{bmatrix} c_n \\ d_n \end{bmatrix} = \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

From example:

$$r(t) = \frac{4}{\pi} \cdot \frac{1}{n^2} \cos(nt), \quad n = 1, 3, 5, \ldots$$

$$a_0 = 0 \Rightarrow c_0 = 0 \qquad a_n = \frac{4}{\pi n^2}, \; b_n = 0$$

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$$\begin{bmatrix} k-n^2 & cn \\ -cn & k-n^2 \end{bmatrix} \begin{bmatrix} c_n \\ d_n \end{bmatrix} = \begin{bmatrix} \frac{4}{\pi n^2} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_n \\ d_n \end{bmatrix} = \frac{1}{(k-n^2)^2 + c^2n^2} \begin{bmatrix} k-n^2 & -cn \\ cn & k-n^2 \end{bmatrix} \begin{bmatrix} \frac{4}{\pi n^2} \\ 0 \end{bmatrix}$$

$$= \frac{1}{(k-n^2)^2 + c^2n^2} \begin{bmatrix} \frac{4}{\pi n^2}(k-n^2) \\ \frac{4}{\pi n^2} cn \end{bmatrix} = \frac{4}{\pi n^2\left[(k-n^2)^2 + c^2n^2\right]} \begin{bmatrix} k-n^2 \\ cn \end{bmatrix}$$

$$y_n = c_n \cos nt + d_n \sin nt \qquad y = y_1 + y_3 + y_5 + \cdots$$

$$c_n = \frac{4}{\pi n^2 D_n}(k-n^2) \;,\; d_n = \frac{4}{\pi n^2 D_n} cn \;,\; D_n = (k-n^2)^2 + c^2n^2$$

Looking at the amplitude of $y_n$ we have

$$|y_n| = \sqrt{c_n^2 + d_n^2} = 4/\pi n^2\sqrt{D_n}$$

| | $\lvert y_1\rvert$ | $\lvert y_3\rvert$ | $\lvert y_5\rvert$ | $\lvert y_7\rvert$ | $\lvert y_9\rvert$ |
|---|---|---|---|---|---|
| $k=25,\ c=0.05$ | 0.053052 | 0.0088416 | 0.20372 | 0.0010826 | 0.00028069 |
| $k=49,\ c=0.05$ | 0.026526 | 0.0035368 | 0.002122 | 0.074241 | 0.00049117 |
| $k=25,\ c=0.5$ | 0.05304 | 0.0088033 | 0.020372 | 0.0010714 | 0.00027979 |
| $k=49,\ c=0.5$ | 0.026524 | 0.0035343 | 0.0021106 | 0.0074241 | 0.00048643 |

Increasing k will shift the peak to the right

Increasing c will reduce the peak amplitude

Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with $r(t)$ as given. Show the details of your work.

⑥ $r(t) = \sin \alpha t + \sin \beta t$ , $\omega^2 \neq \alpha^2, \beta^2$

$$y(t) = y_h(t) + y_p(t) \qquad s^2 + \omega^2 = 0 \Rightarrow s = \pm i\omega$$
$$= c_1 \cos \omega t + c_2 \sin \omega t + y_p(t)$$

$$y_p'' + \omega^2 y_p = r(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$y_p = C_0 + \sum_{n=1}^{\infty} (C_n \cos nt + d_n \sin nt) \qquad y_p' = \sum_{n=1}^{\infty} (-nC_n \sin nt + n d_n \cos nt)$$
$$y_p'' = \sum_{n=1}^{\infty} (-n^2 C_n \cos nt - n^2 d_n \sin nt)$$

$$\sum_{n=1}^{\infty} (-n^2 C_n \cos nt - n^2 d_n \sin nt) + \omega^2 C_0 + \omega^2 \sum_{n=1}^{\infty} (C_n \cos nt + d_n \sin nt) =$$
$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$\omega^2 C_0 = a_0 \Rightarrow C_0 = a_0 / \omega^2$$
$$C_n (\omega^2 - n^2) = a_n \Rightarrow C_n = a_n / (\omega^2 - n^2)$$
$$d_n (\omega^2 - n^2) = b_n \Rightarrow d_n = b_n / (\omega^2 - n^2)$$

Since $r(t) = \sin \alpha t + \sin \beta t$ $\quad a_0 = 0$, $a_n = 0$ for all n

$b_\alpha = 1$ , $b_\beta = 1$ , $b_m = 0$

$$C_\alpha = 0, \; d_\alpha = 1/(\omega^2 - \alpha^2), \; C_\beta = 0, \; d_\beta = 1/(\omega^2 - \beta^2)$$

$$\boxed{y(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{1}{(\omega^2 - \alpha^2)} \sin \alpha t + \frac{1}{(\omega^2 - \beta^2)} \sin \beta t}$$

Sec. 11.4 p 498

Find the trigonometric polynomial $F(x)$ of the form (2) for which the square error with respect to the given $f(x)$ on the interval $-\pi < x < \pi$ is minimum. Compute the minimum value for $N = 1, 2, \dots, 5$ (or also for larger values if you have a CAS).

8. $f(x) = |\sin x|$ $(-\pi < x < \pi)$, full-wave rectifier

$$A_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -\sin x\,dx + \int_0^{\pi} \sin x\,dx\right] = \frac{1}{2\pi}\left[\cos x \Big|_{-\pi}^{0} - \cos x \Big|_0^{\pi}\right]$$
$$= \frac{1}{2\pi}\left[\cos 0 - \cos -\pi - \cos\pi + \cos 0\right] = \frac{1}{2\pi}\left[2-(-2)\right] = \frac{2}{\pi}$$

$$A_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin x \cos nx\,dx + \int_0^{\pi} \sin x \cos nx\,dx\right] = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

$u = \sin x$ $\quad dv = \cos nx\,dx$

$du = \cos x\,dx$ $\quad v = \frac{1}{n}\sin nx$

$$A_n = \frac{2}{\pi}\left[\cancelto{0}{\frac{1}{n}\sin x \sin nx}\Big|_0^{\pi} - \int_0^{\pi} \frac{1}{n}\sin nx \cos x\,dx\right] = \frac{-2}{\pi n}\int_0^{\pi} \cos x \sin nx\,dx$$

$u = \cos x$ $\quad dv = \sin nx\,dx$

$du = -\sin x\,dx$ $\quad v = -\frac{1}{n}\cos nx$

$$A_n = \frac{-2}{\pi n}\left[-\frac{1}{n}\cos x \cos nx\Big|_0^{\pi} - \int_0^{\pi} \frac{1}{n}\sin x \cos nx\,dx\right]$$
$$= \frac{+2}{\pi n^2}\cos x \cos nx\Big|_0^{\pi} + \frac{2}{\pi n^2}\int_0^{\pi} \sin x \cos nx\,dx$$
$$= \frac{2}{\pi n^2}\cos x \cos nx\Big|_0^{\pi} + \frac{1}{n^2}A_n \qquad A_n - \frac{1}{n^2}A_n = \frac{2}{\pi n^2}\cos x \cos nx\Big|_0^{\pi}$$

$$A_n = \frac{2}{\pi(n^2-1)}\cos x \cos nx\Big|_0^{\pi} = \frac{2}{\pi(n^2-1)}\left[\cos\pi \cos n\pi - \cos 0 \cos 0\right]$$
$$= \frac{2}{\pi(n^2-1)}\left[-1(-1)^n - 1\right] = \frac{-2}{\pi(n^2-1)}\left[(-1)^n + 1\right]$$

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$B_n = 0$ because $f(x)$ is even and $f(x)\sin(nx)$ is odd.

$$F(x) = \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{1}{n^2-1} \left[(-1)^n + 1\right] \cos nx$$

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi \left[ 2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right]$$

$$\int_{-\pi}^{\pi} |\sin x|^2 dx = \int_{-\pi}^{0} (-\sin x)^2 dx + \int_{0}^{\pi} \sin^2 x \, dx = \int_{-\pi}^{\pi} \sin^2 x \, dx$$

$$= \left[ \frac{x}{2} - \frac{1}{4} \sin 2x \right]_{-\pi}^{\pi} = \left[ \frac{\pi}{2} - \frac{1}{4} \overset{0}{\cancel{\sin 2\pi}} \right] - \left[ \frac{-\pi}{2} - \frac{1}{4} \overset{0}{\cancel{\sin -2\pi}} \right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$E^* = \pi - \pi \left[ 2\left(\frac{2}{\pi}\right)^2 + \sum_{n=1}^{N} \left( \frac{-2}{\pi(n^2-1)} \left[(-1)^n + 1\right] \right)^2 \right]$$

$$= \pi - \pi \left[ \frac{8}{\pi^2} + \frac{4}{\pi^2} \sum_{n=1}^{N} \frac{1}{(n^2-1)^2} \left[(-1)^n + 1\right]^2 \right] = \pi - \left[ \frac{8}{\pi} + \frac{4}{\pi} \sum_{n=1}^{N} \frac{1}{(n^2-1)^2} \left((-1)^n + 1\right)^2 \right]$$

| N | E* |
|---|---|
| 1 | N/A |
| 2 | 0.029229 |
| 3 | 0.029229 |
| 4 | 0.006594 |
| 5 | 0.006594 |
| 6 | 0.0024364 |
| 7 | 0.0024364 |

Using (8), prove that the series has the indicated sum. Compute the first few partial sums to see that the convergence is rapid.

11. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} = 1.233700550$

Use Example 1 in Sec. 11.1.

From example $f(x) \approx \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin((2n+1)x)$

(8) $2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx$ $\quad f(x)^2 = 1$ for all $x$

$$\sum_{n=0}^{\infty} \left(\frac{4}{\pi} \cdot \frac{1}{2n+1}\right)^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \, dx = \frac{1}{\pi}[\pi + \pi] = 2$$

$$\frac{16}{\pi^2} \cdot \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = 2$$

$$\frac{16}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right] = 2$$

$$\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right] = 2 \cdot \frac{\pi^2}{16}$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8} \checkmark$$

⑫ $1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots=\frac{\pi^4}{90}=1.08232324$

Use Prob. 14 in Sec. 11.1.

From Prob. 14 $\quad f(x)\approx\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos(nx)\approx x^2$

(8) $\quad 2a_0^2+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)^2dx$

$$2\left(\frac{\pi^2}{3}\right)^2+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n\right)^2=\frac{1}{\pi}\int_{-\pi}^{\pi}x^4dx=\frac{1}{\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[\frac{\pi^5}{5}+\frac{\pi^5}{5}\right]$$

$$\frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\left(\frac{1}{n^2}(-1)^n\right)^2=\frac{2}{5}\pi^4$$

$$16\left[1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right]=\frac{2}{5}\pi^4-\frac{2}{9}\pi^4$$

$$\left[1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right]=\frac{1}{16}\cdot\left(\frac{8}{45}\right)\pi^4$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots=\frac{\pi^4}{90}\ \checkmark$$

Sec. 11.5 p 503

(5.) Legendre polynomials. Show that the functions $P_n(\cos\theta)$, $n = 0, 1, \dots$, form an orthogonal set on the interval $0 \le \theta \le \pi$ with respect to the weight function $\sin\theta$.

$P_n(x)$ = Legendre polynomial of degree n

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0 \quad \text{if } n \neq m$$

Change of variables: $x = \cos\theta$ $\quad x = 1 \Rightarrow \theta = 0$

$dx = -\sin\theta\, d\theta \quad x = -1 \Rightarrow \theta = \pi$

$$\int_{\pi}^{0} -P_n(\cos\theta) P_m(\cos\theta) \sin\theta\, d\theta = \int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta\, d\theta \checkmark$$

Find the eigenvalues and eigenfunctions. Verify orthogonality. Start by writing the ODE in the form (1), using Prob. 6. Show details of your work.

(7.) $y'' + \lambda y = 0$, $y(0) = 0$, $y(10) = 0$

$p(x) = 1$, $q(x) = 0$, $r(x) = 1$

Case 1: $\lambda = -\nu^2 < 0$ $(\nu > 0)$

$0 = s^2 + \lambda = s^2 - \nu^2 \Rightarrow s = \pm\nu$, $e^{-\nu x}$, $e^{\nu x}$

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$$y(x) = c_1 e^{-\nu x} + c_2 e^{\nu x}$$

$$0 = y(0) = c_1 + c_2$$

$$0 = y(10) = c_1 e^{-10\nu} + c_2 e^{10\nu} \Rightarrow \begin{bmatrix} 1 & 1 \\ e^{-10\nu} & e^{10\nu} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore\ c_1 = c_2 = 0$$

Case 2 : $\lambda = 0$

$$y'' = 0 \Rightarrow y' = c_2 \Rightarrow y = c_1 + c_2 x$$

$$0 = y(0) = c_1 + c_2(0) \Rightarrow c_1 = 0$$

$$0 = y(10) = c_1 + 10c_2 \Rightarrow c_2 = 0$$

Case 3 : $\lambda = \nu^2\ (\nu > 0)$

$$0 = s^2 + \lambda = s^2 + \nu^2 \Rightarrow s = \pm \nu i$$

$$y(x) = c_1 \cos \nu x + c_2 \sin \nu x$$

$$0 = y(0) = c_1 \cos 0 + c_2 \sin 0 \Rightarrow c_1 = 0$$

$$0 = y(10) = c_1 \cos \nu 10 + c_2 \sin \nu 10 \Rightarrow c_2 \sin \nu 10 = 0$$

$$\sin \nu 10 = 0 \qquad \nu = k\pi/10\ , \ k = 1, 2, \ldots$$

$$\boxed{\begin{array}{l} \lambda = (k\pi/10)^2,\ k = 1, 2, \ldots \\ v(x) = \sin\left(\frac{k\pi}{10} x\right),\ k = 1, 2, \ldots \end{array}}$$

NOTE: Only one eigenvalue / function hence orthogonal

⑬ $y'' + 8y' + (\lambda + 16)y = 0$ , $y(0) = 0$ , $y(\pi) = 0$

$p(x) = 8$ , $q(x) = 16$ , $r(x) = 1$

Case 1 : $\lambda = -\nu^2$ $(\nu > 0)$

$0 = s^2 + 8s + \lambda + 16 = s^2 + 8s - \nu^2 + 16 = s^2 + 8s + 16 - \nu^2$

$$s = \frac{-8 \pm \sqrt{8^2 - 4(16 - \nu^2)}}{2} = \frac{-8 \pm \sqrt{64 - \cancel{64} + 4\nu^2}}{2} = -4 \pm \nu$$

$y(x) = c_1 e^{(-4+\nu)x} + c_2 e^{(-4-\nu)x}$

$y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = 0$

$y(\pi) = c_1 e^{-4\pi} e^{\nu\pi} + c_2 e^{-4\pi} e^{-\nu\pi} = c_1 e^{4\pi} + c_2 e^{-4\pi} = 0$

$$\begin{bmatrix} 1 & 1 \\ e^{+4\nu} & e^{-4\nu} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 1 \\ e^{+4\nu} & e^{-4\nu} \end{bmatrix} = e^{-4\nu} + e^{4\nu} \neq 0 \quad \therefore \quad \begin{matrix} c_1 = 0 \\ c_2 = 0 \end{matrix}$$

Case 2 : $\lambda = 0$

$0 = s^2 + 8s + 16 \Rightarrow (s+4)(s+4) = 0$

$y(x) = c_1 e^{-4x} + x c_2 e^{-4x}$

$y(0) = c_1 e^0 + (0) c_2 e^0 \Rightarrow c_1 = 0$

$y(\pi) = c_1 e^{-4\pi} + \pi c_2 e^{-4\pi} \Rightarrow \pi c_2 e^{-4\pi} = 0 \quad c_2 = 0$

Case 3 : $\lambda = \nu^2$ $(\nu > 0)$

$0 = s^2 + 8s + \lambda + 16 = s^2 + 8s + \nu^2 + 16$

$$s = \frac{-8 \pm \sqrt{8^2 - 4(16 + \nu^2)}}{2} = \frac{-8 \pm \sqrt{64 - 64 - 4\nu^2}}{2} = -4 \pm i\nu$$

$y(x) = c_1 e^{-4x} \cos \nu x + c_2 e^{-4x} \sin \nu x$

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$$y(0) = c_1 e^{0} \cos 0 + c_2 e^{0} \sin 0 \Rightarrow c_1 = 0$$

$$y(\pi) = c_1 e^{-4\pi} \cos \sqrt{\pi} + c_2 e^{-4\pi} \sin \sqrt{\pi} \Rightarrow c_2 e^{-4\pi} \sin \sqrt{\pi} = 0$$

$$\sin \sqrt{\pi} = 0 \Rightarrow \sqrt{} = k \;,\; k = 0, 1, 2, \ldots$$

$$\lambda = k^2 \;,\; v(x) = e^{-4x} \sin kx \;,\; k = 0, 1, 2, \ldots$$

$$\boxed{\begin{matrix} \lambda = k^2 \\ v(x) = e^{-4x} \sin kx \end{matrix} \;,\; k = 0, 1, 2, \ldots}$$