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11. Solution:

This is an even function.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{3}$$

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$$a_n = 1\cdot\int_{-1}^{1} x^2 \cos\frac{n\pi x}{1}\,dx = 2\int_0^1 x^2\cos n\pi x\,dx$$
$$= \frac{2}{n\pi}\int_0^1 x^2\,d[\sin(n\pi x)]$$
$$= \frac{2}{n\pi}x^2\sin(n\pi x)\Big|_0^1 - \frac{4}{n\pi}\int_0^1 x\sin(n\pi x)\,dx$$
$$= \frac{4}{(n\pi)^2}\int_0^1 x\,d[\cos(n\pi x)]$$
$$= \frac{4}{(n\pi)^2}x\cos(n\pi x)\Big|_0^1 - \frac{4}{(n\pi)^2}\int_0^1 \cos(n\pi x)\,dx$$
$$= \frac{4}{(n\pi)^2}\cos n\pi$$
$$= \frac{4}{(n\pi)^2}\cdot(-1)^n$$

$$b_n = 1\cdot\int_{-1}^{1} x^2\sin\frac{n\pi x}{1}\,dx = 0$$

Therefore:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right)$$
$$= \frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{(n\pi)^2}(-1)^n\cos n\pi x\right]$$
$$= \frac{1}{3} + \frac{4}{\pi^2}\left(-\cos\pi x + \frac{1}{4}\cos 2\pi x - \frac{1}{9}\cos 3\pi x - + \cdots\right)$$

20. Proof: From Prob. 11, we have:

$$f(x) = x^2 = \frac{1}{3} + \frac{4}{\pi^2}\left(-\cos\pi x + \frac{1}{4}\cos 2\pi x - \frac{1}{9}\cos 3\pi x - + \cdots\right)$$

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Let $x=1$, then:

$$1 = \frac{1}{3} + \frac{4}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right)$$
$$\Rightarrow 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

24. Solution:

(a) Fourier cosine series.

$$a_0 = \frac{1}{4}\int_0^4 f(x)\,dx = \frac{1}{4}\cdot 2 = \frac{1}{2}$$
$$a_n = \frac{2}{4}\int_0^4 f(x)\cos\frac{n\pi x}{4}\,dx = \frac{1}{2}\int_2^4 \cos\frac{n\pi}{4}x\,dx$$
$$= -\frac{2}{n\pi}\sin\frac{n\pi}{2}$$
$$\Rightarrow a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{L}x = \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi}{4}x\right)$$

$$= \frac{2}{(n-1)(n+1)\pi}\left[(-1)^{n+1}-1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0.$$

$$\Rightarrow F(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{2}{(n-1)(n+1)\pi}\left[(-1)^{n+1}-1\right]\cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{1\times3}\cos x + \frac{1}{3\times5}\cos 3x + \frac{1}{5\times7}\cos 5x + \cdots\right).$$

$$E^* = \int_{-\pi}^{\pi} f(x)^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right]$$

$$= \pi - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2+b_n^2)\right]$$

$$\Rightarrow E^* = 0.59511,\ 0.02923,\ 0.02923,\ 0.00659,\ 0.00659.$$

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11. Proof: According to Example 1 in Section 11.1, we have:

$$a_0 = 0,\quad a_n = 0$$

$$b_n = \frac{2k}{n\pi}(1-\cos n\pi)$$

$$\Rightarrow b_1 = \frac{4k}{\pi},\ b_2 = 0,\ b_3 = \frac{4k}{3\pi},\ b_4 = 0,\ b_5 = \frac{4k}{5\pi}.\ \cdots$$

Also, $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = 2k^2$

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Using (8), we know:

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx$$

$$\Rightarrow \left(\frac{4k}{\pi}\right)^2 + \left(\frac{4k}{3\pi}\right)^2 + \left(\frac{4k}{5\pi}\right)^2 + \cdots = \cancel{4}\,2k^2$$

$$\Rightarrow 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} = 1.2337$$

First few partial sums:

$1 + \frac{1}{3^2} = 1.111111$ $\quad E^* = 0.1225894$

$1 + \frac{1}{3^2} + \frac{1}{5^2} = 1.151111$ $\quad E^* = 0.0825894$

$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} = 1.1715127$ $\quad E^* = 0.0621828$

12. Proof: According to Prob. 14 in Sec. 11.1, $f(x) = x^2.\ (-\pi < x < \pi)$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx.$$

$$= \frac{2}{n\pi}\int_0^{\pi} x^2\,d(\sin nx)$$

$$= \frac{2}{n\pi}x^2\sin nx\Big|_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi}\sin nx\cdot x\,dx$$

$$= \frac{4}{n^2\pi}\int_0^{\pi} x\,d(\cos nx)$$

$$= \frac{4}{n^2\pi}x\cos nx\Big|_0^{\pi} - \frac{4}{n^2\pi}\int_0^{\pi}\cos nx\,dx$$

$$= \frac{4}{n^2}(-1)^n$$

Using (8) we have:

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx$$

$$\Rightarrow 2\cdot(\tfrac{1}{3}\pi^2)^2 + \sum_{n=1}^{\infty}(\tfrac{4}{n^2})^2 = \tfrac{2}{5}\pi^4$$

$$\Rightarrow 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} = 1.082323\cdots$$

First few partial sums:

$1 + \frac{1}{2^4} = 1.0625$ $\qquad E^* = 0.019823$

$1 + \frac{1}{2^4} + \frac{1}{3^4} = 1.0748$ $\qquad E^* = 0.007478$

$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} = 1.0788$ $\qquad E^* = 0.003571$

P503. 5. Proof:

$$f(P_m(\cos\theta), P_n(\cos\theta)) = \int_0^{\pi} \sin\theta\, P_m(\cos\theta)\, P_n(\cos\theta)\, d\theta$$

$$= -\int_0^{\pi} P_m(\cos\theta)\, P_n(\cos\theta)\, d(\cos\theta)$$

Let $x = \cos\theta$, then the integral becomes:

$$-\int_1^{-1} P_m(x)\, P_n(x)\, dx$$

$$= \int_{-1}^{1} P_m(x)\, P_n(x)\, dx$$

$$= 0$$

This completes the proof.

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7. Solution:

Writing the ODE in the form (1):

$$y'' + \lambda y = 0 \qquad (p=1,\ q=0,\ r=1)$$

For negative $\lambda = -v^2$ a general solution of ODE is $y(x) = c_1 e^{vx} + c_2 e^{-vx}$. From the boundary condition we obtain $c_1 = c_2 = 0$, so $y \equiv 0$, which is not eigenfunction.

For positive $\lambda = v^2$ a general solution is:

$$y(x) = A\cos vx + B\sin vx$$

From $y(0) = y(10) = 0$, we have:

$$\begin{cases} A = 0 \\ B\sin 10v = 0 \end{cases}$$

$$\Rightarrow 10v = n\pi$$

$$\Rightarrow v = \frac{n\pi}{10}$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{10}\right)^2,$$

For $n=0$, we have $v=0$, $y \equiv 0$. So $n \neq 0$.