#5 $A = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -3 & -6 & 0 \end{pmatrix}$ $r(A) = ?$

$$\longrightarrow \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$r(A) = 1$

B

#6

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 6 & 3 \end{pmatrix} \qquad \lambda(A) = ?$$

$$0 = \begin{vmatrix} 1-\lambda & 0 & 1 \\ \cancel{0} & \cancel{2-\lambda} & \cancel{0} \\ 3 & 6 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 3 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\Big[\underbrace{(\lambda-1)(\lambda-3) - 3}_{\lambda^2 - 4\lambda \,=\, \lambda(\lambda-4)} \Big]$$

$\lambda = 0, 2, 4$

E

#7

$$A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \underline{\lambda(A) = 2, 0, -1}$$

$$\underline{A\vec{x} = \lambda\vec{x}}$$

$$\vec{x} \neq \vec{0}$$

C

$$A+I = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 - x_2 = 0 \\ x_2 - x_3 = 0 \end{cases} \qquad \underline{x_2 = s} \qquad \vec{x} = \begin{pmatrix} s \\ s \\ s \end{pmatrix} = \underline{\underline{s}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

diagonalize $A_{2\times 2}$

λ_1, λ_2

$\vec{x}_1, \vec{x}_2$

$\lambda_1 = \lambda_2$

$$X = [\vec{x}_1, \vec{x}_2]$$

$$AX = X \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$X^{-1}AX = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

#8 $\vec{y}\,' = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \vec{y}$, $\vec{y}_1 = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$, $\vec{y}_2 = ?$

$$0 = \begin{vmatrix} 3-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda-3)(\lambda-2) \Longrightarrow \lambda_1 = 2,\ \lambda_2 = 3$$

$\lambda_1 = 2$ $\quad x_1 = 0 \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \vec{y}_1 = e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_2 = 3$ $\quad \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad x_1 = x_2 \quad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\vec{y}_2 = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{vmatrix} 0 & e^{3t} \\ e^{2t} & e^{2t} + e^{3t} \end{vmatrix} = -e^{5t} \neq 0$$

E. $e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

#9 $\vec{y}' = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \vec{y}$ (0,0) —

$$0 = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = (\lambda-1)(\lambda-2) - 12$$

$$= \lambda^2 - 3\lambda - 10$$

$$\lambda = \frac{3 \pm \sqrt{9+40}}{2} = \frac{3 \pm 7}{2}$$

$= -2, 5$ — saddle pt

real — node

complex — $\pm i\mu$ — center; spiral $\lambda \pm i\mu$

$\lambda > 0$ unstable

$\lambda < 0$ stable

$$a^2 - b^2 = (a+b)(a-b)$$

#10 $\vec{y}' = \begin{pmatrix} 6 & 9 \\ 1 & 6 \end{pmatrix} \vec{y}$ (0, 0)

$$0 = \begin{vmatrix} 6-\lambda & 9 \\ 1 & 6-\lambda \end{vmatrix} = (\lambda-6)^2 - 9 = (\lambda-3)(\lambda-9)$$

$\lambda = 3, 9$ node

unstable

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

critical pts

$$\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases}$$

(x_0, y_0)

$$\boxed{\vec{\nabla} f(x_0,y_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}_{(x_0,y_0)}}$$

#11

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \dfrac{3xy}{1+x^2+y^2} - \dfrac{1+x^2}{1+y^2} \\ x^2 - y^2 \end{pmatrix} \qquad \underline{(1,1)}$$

$$0 - 2y(1+x^2)$$

$$\frac{\partial f_1}{\partial \overset{}{\underset{y}{x}}}^{(1,1)} = \frac{3\cancel{y}x(1+x^2+y^2) - 2\overset{y}{\cancel{x}}\cdot 3xy}{(1+x^2+y^2)^2} - \left.\frac{2x(1+y^2) - 0}{(1+y^2)^2}\right|_{(1,1)} \qquad \frac{\partial f_1}{\partial y}(1,1) = \frac{1}{3} + 1 = \frac{4}{3}$$

$$= \frac{9-6}{9} - \frac{4}{4} = \frac{1}{3} - 1 = -\frac{2}{3}$$

$$\frac{\partial f_2}{\partial x}(1,1) = 2x\Big|_{(1,1)} = 2, \quad \frac{\partial f_2}{\partial y} = -2y \to -2$$

$$\begin{vmatrix} -\frac{2}{3} - \lambda & \frac{4}{3} \\ 2 & -2-\lambda \end{vmatrix} = \left(\lambda + \frac{2}{3}\right)(\lambda+2) - \frac{8}{3} = \lambda^2 + \frac{8}{3}\lambda - \frac{4}{3}$$

$$\nabla f(1,1) = \begin{pmatrix} -\frac{2}{3} & \frac{4}{3} \\ 2 & -2 \end{pmatrix}$$

$$0 = |A - \lambda I| = \lambda^2 + \frac{8}{3}\lambda - \frac{4}{3}$$

$$\lambda = \frac{-\frac{8}{3} \pm \sqrt{\left(\frac{8}{3}\right)^2 + \frac{16}{3}}}{2}$$

real distict ~~node~~

$\lambda_1 < 0$ $\lambda_2 > 0$

saddle pt.

unstable.

#12 $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ \sin x \end{pmatrix}$

$\boxed{\begin{cases} y = 0 \\ \sin x = 0 \quad x = k\pi \end{cases}}$

(0, 0)

$$\nabla f(0,0) = \begin{pmatrix} 0 & 1 \\ \cos x & 0 \end{pmatrix}_{(0,0)}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$0 = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 \Rightarrow \lambda = \pm 1$$

saddle pt
unstable

#13

$$\vec{x}' = A\vec{x} + \begin{pmatrix} 2e^{-t} \\ 2 \end{pmatrix} \qquad \vec{u}' = X^{-1}\vec{g}$$

$$X(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} \qquad X^{-1} = \frac{1}{2e^{-4t}} \begin{pmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix}$$

$$\vec{u}' = \frac{1}{2e^{-4t}} \begin{pmatrix} 2e^{-2t} - 2e^{-t} \\ 2e^{-4t} + 2e^{-3t} \end{pmatrix} = \begin{pmatrix} e^{2t} - e^{3t} \\ 1 + e^{t} \end{pmatrix}$$

$$\vec{u} = \begin{pmatrix} \frac{1}{2}e^{2t} - \frac{1}{3}e^{3t} \\ t + e^{t} \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad X\vec{u}$$

$$= \begin{pmatrix} \frac{1}{2}e^{-t} + te^{-t} + \frac{2}{3} \\ -\frac{1}{2}e^{-t} + te^{-t} + \frac{4}{3} \end{pmatrix}$$

$$\vec{x} = X\vec{u} = X\vec{c} + \underline{X\begin{pmatrix} \frac{1}{2}e^{2t} - \frac{1}{3}e^{3t} \\ t + e^{t} \end{pmatrix}}$$

$$= \begin{pmatrix} \frac{1}{2}e^{-t} - \frac{1}{3} + te^{-t} + 1 \\ -\frac{1}{2}e^{-t} + \frac{1}{3} + te^{-t} + 1 \end{pmatrix}$$

Terry Loro

Undergrad Service Office

Math 242

loro@math.purdue.edu

765-494-1991

Off-Campus

Pro Ed