

PROBLEM SET 11.1

1-5 PERIOD, FUNDAMENTAL PERIOD

The *fundamental period* is the smallest positive period. Find it for

- $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$
- $\cos nx$, $\sin nx$, $\cos \frac{2\pi x}{k}$, $\sin \frac{2\pi x}{k}$, $\cos \frac{2\pi nx}{k}$, $\sin \frac{2\pi nx}{k}$
- If $f(x)$ and $g(x)$ have period p , show that $h(x) = af(x) + bg(x)$ (a, b , constant) has the period p . Thus all functions of period p form a **vector space**.
- Change of scale.** If $f(x)$ has period p , show that $f(ax)$, $a \neq 0$, and $f(x/b)$, $b \neq 0$, are periodic functions of x of periods p/a and bp , respectively. Give examples.
- Show that $f = \text{const}$ is periodic with any period but has no fundamental period.

6-10 GRAPHS OF 2π -PERIODIC FUNCTIONS

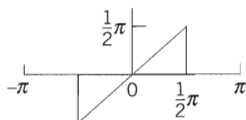
Sketch or graph $f(x)$ which for $-\pi < x < \pi$ is given as follows.

- $f(x) = |x|$
- $f(x) = |\sin x|$, $f(x) = \sin |x|$
- $f(x) = e^{-|x|}$, $f(x) = |e^{-x}|$
- $f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$
- $f(x) = \begin{cases} -\cos^2 x & \text{if } -\pi < x < 0 \\ \cos^2 x & \text{if } 0 < x < \pi \end{cases}$
- Calculus review.** Review integration techniques for integrals as they are likely to arise from the Euler formulas, for instance, definite integrals of $x \cos nx$, $x^2 \sin nx$, $e^{-2x} \cos nx$, etc.

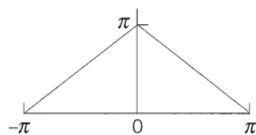
12-21 FOURIER SERIES

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π . Show the details of your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.

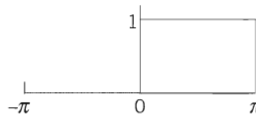
- $f(x)$ in Prob. 6
- $f(x)$ in Prob. 9
- $f(x) = x^2$ ($-\pi < x < \pi$)
- $f(x) = x^2$ ($0 < x < 2\pi$)
- 16.



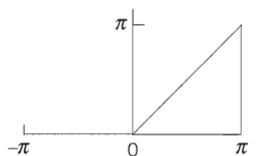
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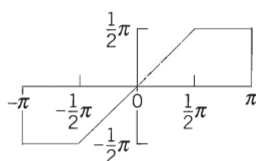
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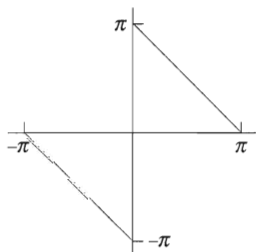
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20.



21.



22. CAS EXPERIMENT. Graphing. Write a program for graphing partial sums of the following series. Guess from the graph what $f(x)$ the series may represent. Confirm or disprove your guess by using the Euler formulas.

- $2(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots)$
 $- 2(\frac{1}{2} \sin 2x + \frac{1}{4} \sin 4x + \frac{1}{6} \sin 6x + \dots)$
- $\frac{1}{2} + \frac{4}{\pi^2} (\cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \dots)$
- $\frac{2}{3} \pi^2 + 4(\cos x - \frac{1}{4} \cos 2x + \frac{1}{9} \cos 3x - \frac{1}{16} \cos 4x + \dots)$

23. Discontinuities. Verify the last statement in Theorem 2 for the discontinuities of $f(x)$ in Prob. 21.

24. CAS EXPERIMENT. Orthogonality. Integrate and graph the integral of the product $\cos mx \cos nx$ (with various integer m and n of your choice) from $-a$ to a as a function of a and conclude orthogonality of $\cos nx$.

and $\cos nx$ ($m \neq n$) for $a = \pi$ from the graph. For what m and n will you get orthogonality for $a = \pi/2, \pi/3, \pi/4$? Other a ? Extend the experiment to $\cos mx \sin nx$ and $\sin mx \sin nx$.

25. **CAS EXPERIMENT. Order of Fourier Coefficients.** The order seems to be $1/n$ if f is discontinuous, and $1/n^2$

if f is continuous but $f' = df/dx$ is discontinuous, $1/n^3$ if f and f' are continuous but f'' is discontinuous, etc. Try to verify this for examples. Try to prove it by integrating the Euler formulas by parts. What is the practical significance of this?

11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions

We now expand our initial basic discussion of Fourier series.

Orientation. This section concerns three topics:

1. Transition from period 2π to any period $2L$, for the function f , simply by a transformation of scale on the x -axis.
2. Simplifications. Only cosine terms if f is even ("Fourier cosine series"). Only sine terms if f is odd ("Fourier sine series").
3. Expansion of f given for $0 \leq x \leq L$ in two Fourier series, one having only cosine terms and the other only sine terms ("half-range expansions").

1. From Period 2π to Any Period $p = 2L$

Clearly, periodic functions in applications may have any period, not just 2π as in the last section (chosen to have simple formulas). The notation $p = 2L$ for the period is practical because L will be a length of a violin string in Sec. 12.2, of a rod in heat conduction in Sec. 12.5, and so on.

The transition from period 2π to be period $p = 2L$ is effected by a suitable change of scale, as follows. Let $f(x)$ have period $p = 2L$. Then we can introduce a new variable v such that $f(x)$, as a function of v , has period 2π . If we set

$$(1) \quad (a) \quad x = \frac{p}{2\pi} v, \quad \text{so that} \quad (b) \quad v = \frac{2\pi}{p} x = \frac{\pi}{L} x$$

then $v = \pm\pi$ corresponds to $x = \pm L$. This means that f , as a function of v , has period 2π and, therefore, a Fourier series of the form

$$(2) \quad f(x) = f\left(\frac{L}{\pi} v\right) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nv + b_n \sin nv)$$

with coefficients obtained from (6) in the last section

$$(3) \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) dv, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \cos nv \, dv,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi} v\right) \sin nv \, dv.$$

Write a program for finding series. Guesses may represent y using the Euler

\dots)
 $4x + \frac{1}{6} \sin 6x \dots$
 $\frac{1}{25} \cos 5x + \dots$
 $\cos 3x - \frac{1}{16} \cos 4x$
 $+ \dots$)

statement in Theorem Prob. 21.

Property. Integrate and $\cos mx \cos nx$ (with \cos in place of \sin) from $-a$ to a to show orthogonality of $\cos mx$

We insert these two results into the formula for a_n . The sine terms cancel and so does a factor L^2 . This gives

$$a_n = \frac{4k}{n^2\pi^2} \left(2 \cos \frac{n\pi}{2} - \cos n\pi - 1 \right).$$

Thus,

$$a_2 = -16k/(2^2\pi^2), \quad a_6 = -16k/(6^2\pi^2), \quad a_{10} = -16k/(10^2\pi^2), \dots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \dots$. Hence the first half-range expansion of $f(x)$ is (Fig. 272a)

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2} \left(\frac{1}{2^2} \cos \frac{2\pi}{L} x + \frac{1}{6^2} \cos \frac{6\pi}{L} x + \dots \right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.

(b) *Odd periodic extension.* Similarly, from (6**) we obtain

$$(5) \quad b_n = \frac{8k}{n^2\pi^2} \sin \frac{n\pi}{2}.$$

Hence the other half-range expansion of $f(x)$ is (Fig. 272b)

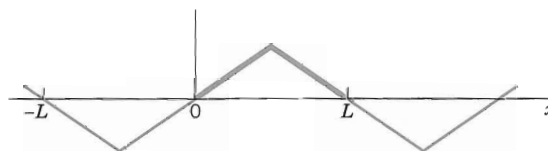
$$f(x) = \frac{8k}{\pi^2} \left(\frac{1}{1^2} \sin \frac{\pi}{L} x - \frac{1}{3^2} \sin \frac{3\pi}{L} x + \frac{1}{5^2} \sin \frac{5\pi}{L} x - \dots \right).$$

The series represents the odd periodic extension of $f(x)$, of period $2L$.

Basic applications of these results will be shown in Secs. 12.3 and 12.5.



(a) Even extension



(b) Odd extension

Fig. 272. Periodic extensions of $f(x)$ in Example 6

PROBLEM SET 11.2

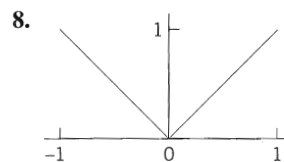
1-7 EVEN AND ODD FUNCTIONS

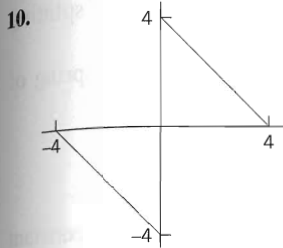
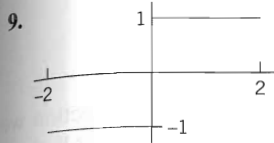
Are the following functions even or odd or neither even nor odd?

- e^x , $e^{-|x|}$, $x^3 \cos nx$, $x^2 \tan \pi x$, $\sinh x - \cosh x$
- $\sin^2 x$, $\sin(x^2)$, $\ln x$, $x/(x^2 + 1)$, $x \cot x$
- Sums and products of even functions
- Sums and products of odd functions
- Absolute values of odd functions
- Product of an odd times an even function
- Find all functions that are both even and odd.

8-17 FOURIER SERIES FOR PERIOD $p = 2L$

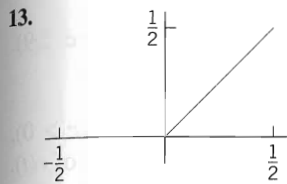
Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.



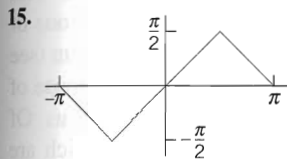


11. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$

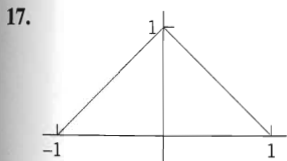
12. $f(x) = 1 - x^2/4 \quad (-2 < x < 2), \quad p = 4$



14. $f(x) = \cos \pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1$



16. $f(x) = x|x| \quad (-1 < x < 1), \quad p = 2$



18. **Rectifier.** Find the Fourier series of the function obtained by passing the voltage $v(t) = V_0 \cos 100\pi t$ through a half-wave rectifier that clips the negative half-waves.

19. **Trigonometric Identities.** Show that the familiar identities $\cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x$ and $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$ can be interpreted as Fourier series expansions. Develop $\cos^4 x$.

20. **Numeric Values.** Using Prob. 11, show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{1}{6} \pi^2$.

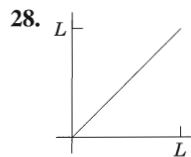
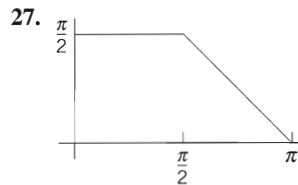
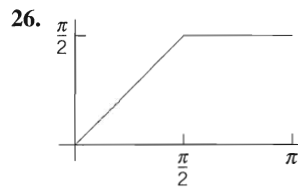
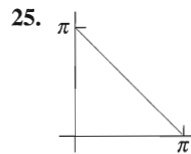
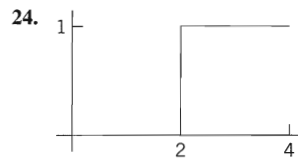
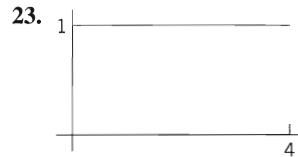
21. **CAS PROJECT. Fourier Series of 2L-Periodic Functions.** (a) Write a program for obtaining partial sums of a Fourier series (5).

(b) Apply the program to Probs. 8–11, graphing the first few partial sums of each of the four series on common axes. Choose the first five or more partial sums until they approximate the given function reasonably well. Compare and comment.

22. Obtain the Fourier series in Prob. 8 from that in Prob. 17.

23–29 HALF-RANGE EXPANSIONS

Find (a) the Fourier cosine series, (b) the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. Show the details.



29. $f(x) = \sin x \quad (0 < x < \pi)$

30. Obtain the solution to Prob. 26 from that of Prob. 27.

ctor L^2 . This gives

(a)

of period $2L$.

IOD $p = 2L$

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11.3 Forced Oscillations

Fourier series have important applications for both ODEs and PDEs. In this section we shall focus on ODEs and cover similar applications for PDEs in Chap. 12. All these applications will show our indebtedness to Euler's and Fourier's ingenious idea of splitting up periodic functions into the simplest ones possible.

From Sec. 2.8 we know that forced oscillations of a body of mass m on a spring of modulus k are governed by the ODE

$$(1) \quad my'' + cy' + ky = r(t)$$

where $y = y(t)$ is the displacement from rest, c the damping constant, k the spring constant (spring modulus), and $r(t)$ the external force depending on time t . Figure 274 shows the model and Fig. 275 its electrical analog, an RLC -circuit governed by

$$(1^*) \quad LI'' + RI' + \frac{1}{C}I = E'(t) \quad (\text{Sec. 2.9}).$$

We consider (1). If $r(t)$ is a sine or cosine function and if there is damping ($c > 0$), then the steady-state solution is a harmonic oscillation with frequency equal to that of $r(t)$. However, if $r(t)$ is not a pure sine or cosine function but is any other periodic function, then the steady-state solution will be a superposition of harmonic oscillations with frequencies equal to that of $r(t)$ and integer multiples of these frequencies. And if one of these frequencies is close to the (practical) resonant frequency of the vibrating system (see Sec. 2.8), then the corresponding oscillation may be the dominant part of the response of the system to the external force. This is what the use of Fourier series will show us. Of course, this is quite surprising to an observer unfamiliar with Fourier series, which are highly important in the study of vibrating systems and resonance. Let us discuss the entire situation in terms of a typical example.

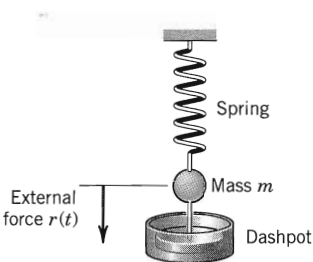


Fig. 274. Vibrating system under consideration

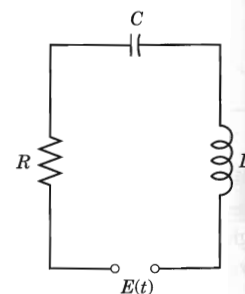


Fig. 275. Electrical analog of the system in Fig. 274 (RLC -circuit)

EXAMPLE 1 Forced Oscillations under a Nonsinusoidal Periodic Driving Force

In (1), let $m = 1$ (g), $c = 0.05$ (g/sec), and $k = 25$ (g/sec²), so that (1) becomes

$$(2) \quad y'' + 0.05y' + 25y = r(t)$$

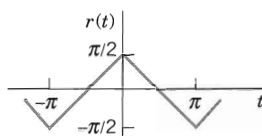


Fig. 276. Force in Example 1

where $r(t)$ is measured in $\text{g} \cdot \text{cm}/\text{sec}^2$. Let (Fig. 276)

$$r(t) = \begin{cases} t + \frac{\pi}{2} & \text{if } -\pi < t < 0, \\ -t + \frac{\pi}{2} & \text{if } 0 < t < \pi, \end{cases} \quad r(t + 2\pi) = r(t).$$

Find the steady-state solution $y(t)$.

Solution. We represent $r(t)$ by a Fourier series, finding

$$(3) \quad r(t) = \frac{4}{\pi} \left(\cos t + \frac{1}{3^2} \cos 3t + \frac{1}{5^2} \cos 5t + \dots \right).$$

Then we consider the ODE

$$(4) \quad y'' + 0.05y' + 25y = \frac{4}{n^2\pi} \cos nt \quad (n = 1, 3, \dots)$$

whose right side is a single term of the series (3). From Sec. 2.8 we know that the steady-state solution $y_n(t)$ of (4) is of the form

$$(5) \quad y_n = A_n \cos nt + B_n \sin nt.$$

By substituting this into (4) we find that

$$(6) \quad A_n = \frac{4(25 - n^2)}{n^2\pi D_n}, \quad B_n = \frac{0.2}{n\pi D_n}, \quad \text{where } D_n = (25 - n^2)^2 + (0.05n)^2.$$

Since the ODE (2) is linear, we may expect the steady-state solution to be

$$(7) \quad y = y_1 + y_3 + y_5 + \dots$$

where y_n is given by (5) and (6). In fact, this follows readily by substituting (7) into (2) and using the Fourier series of $r(t)$, provided that termwise differentiation of (7) is permissible. (Readers already familiar with the notion of uniform convergence [Sec. 15.5] may prove that (7) may be differentiated term by term.)

From (6) we find that the amplitude of (5) is (a factor $\sqrt{D_n}$ cancels out)

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2\pi\sqrt{D_n}}.$$

Values of the first few amplitudes are

$$C_1 = 0.0531 \quad C_3 = 0.0088 \quad C_5 = 0.2037 \quad C_7 = 0.0011 \quad C_9 = 0.0003.$$

Figure 277 shows the input (multiplied by 0.1) and the output. For $n = 5$ the quantity D_n is very small, the denominator of C_5 is small, and C_5 is so large that y_5 is the dominating term in (7). Hence the output is almost a harmonic oscillation of five times the frequency of the driving force, a little distorted due to the term y_1 , whose amplitude is about 25% of that of y_5 . You could make the situation still more extreme by decreasing the damping constant c . Try it. ■

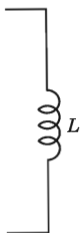
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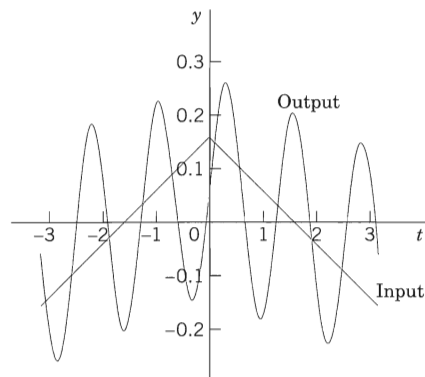


Fig. 277. Input and steady-state output in Example 1

PROBLEM SET 11.3

- Coefficients C_n .** Derive the formula for C_n from A_n and B_n .
- Change of spring and damping.** In Example 1, what happens to the amplitudes C_n if we take a stiffer spring, say, of $k = 49$? If we increase the damping?
- Phase shift.** Explain the role of the B_n 's. What happens if we let $c \rightarrow 0$?
- Differentiation of input.** In Example 1, what happens if we replace $r(t)$ with its derivative, the rectangular wave? What is the ratio of the new C_n to the old ones?
- Sign of coefficients.** Some of the A_n in Example 1 are positive, some negative. All B_n are positive. Is this physically understandable?

6–11 GENERAL SOLUTION

Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with $r(t)$ as given. Show the details of your work.

- $r(t) = \sin \alpha t + \sin \beta t$, $\omega^2 \neq \alpha^2, \beta^2$
- $r(t) = \sin t$, $\omega = 0.5, 0.9, 1.1, 1.5, 10$
- Rectifier.** $r(t) = \pi/4 |\cos t|$ if $-\pi < t < \pi$ and $r(t + 2\pi) = r(t)$, $|\omega| \neq 0, 2, 4, \dots$
- What kind of solution is excluded in Prob. 8 by $|\omega| \neq 0, 2, 4, \dots$?
- Rectifier.** $r(t) = \pi/4 |\sin t|$ if $0 < t < 2\pi$ and $r(t + 2\pi) = r(t)$, $|\omega| \neq 0, 2, 4, \dots$
- $r(t) = \begin{cases} -1 & \text{if } -\pi < t < 0 \\ 1 & \text{if } 0 < t < \pi, \end{cases}$ $|\omega| \neq 1, 3, 5, \dots$
- CAS Program.** Write a program for solving the ODE just considered and for jointly graphing input and output of an initial value problem involving that ODE. Apply

the program to Probs. 7 and 11 with initial values of your choice.

13–16 STEADY-STATE DAMPED OSCILLATIONS

Find the steady-state oscillations of $y'' + cy' + y = r(t)$ with $c > 0$ and $r(t)$ as given. Note that the spring constant is $k = 1$. Show the details. In Probs. 14–16 sketch $r(t)$.

- $r(t) = \sum_{n=1}^N (a_n \cos nt + b_n \sin nt)$
- $r(t) = \begin{cases} -1 & \text{if } -\pi < t < 0 \\ 1 & \text{if } 0 < t < \pi \end{cases}$ and $r(t + 2\pi) = r(t)$
- $r(t) = t(\pi^2 - t^2)$ if $-\pi < t < \pi$ and $r(t + 2\pi) = r(t)$
- $r(t) = \begin{cases} t & \text{if } -\pi/2 < t < \pi/2 \\ \pi - t & \text{if } \pi/2 < t < 3\pi/2 \end{cases}$ and $r(t + 2\pi) = r(t)$

17–19 RLC-CIRCUIT

Find the steady-state current $I(t)$ in the RLC -circuit in Fig. 275, where $R = 10 \Omega$, $L = 1 \text{ H}$, $C = 10^{-1} \text{ F}$ and with $E(t)$ V as follows and periodic with period 2π . Graph or sketch the first four partial sums. Note that the coefficients of the solution decrease rapidly. *Hint.* Remember that the ODE contains $E'(t)$, not $E(t)$, cf. Sec. 2.9.

- $E(t) = \begin{cases} -50t^2 & \text{if } -\pi < t < 0 \\ 50t^2 & \text{if } 0 < t < \pi \end{cases}$

$F = S_1, S_2, S_3$ are shown in Fig. 269 in Sec. 11.2, and $F = S_{20}$ is shown in Fig. 279. Although $|f(x) - F(x)|$ is large at $\pm\pi$ (how large?), where f is discontinuous, F approximates f quite well on the whole interval, except near $\pm\pi$, where “waves” remain owing to the “Gibbs phenomenon,” which we shall discuss in the next section. Can you think of functions f for which E^* decreases more quickly with increasing N ?

PROBLEM SET 11.4

- 1. CAS Problem.** Do the numeric and graphic work in Example 1 in the text.

2-5 MINIMUM SQUARE ERROR

Find the trigonometric polynomial $F(x)$ of the form (2) for which the square error with respect to the given $f(x)$ on the interval $-\pi < x < \pi$ is minimum. Compute the minimum value for $N = 1, 2, \dots, 5$ (or also for larger values if you have a CAS).

2. $f(x) = x \quad (-\pi < x < \pi)$

3. $f(x) = |x| \quad (-\pi < x < \pi)$

4. $f(x) = x^2 \quad (-\pi < x < \pi)$

5. $f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$

6. Why are the square errors in Prob. 5 substantially larger than in Prob. 3?

7. $f(x) = x^3 \quad (-\pi < x < \pi)$

8. $f(x) = |\sin x| \quad (-\pi < x < \pi)$, full-wave rectifier

9. **Monotonicity.** Show that the minimum square error (6) is a monotone decreasing function of N . How can you use this in practice?

10. **CAS EXPERIMENT. Size and Decrease of E^* .** Compare the size of the minimum square error E^* for functions of your choice. Find experimentally the

factors on which the decrease of E^* with N depends. For each function considered find the smallest N such that $E^* < 0.1$.

11-15 PARSEVALS'S IDENTITY

Using (8), prove that the series has the indicated sum. Compute the first few partial sums to see that the convergence is rapid.

11. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8} = 1.233700550$

Use Example 1 in Sec. 11.1.

12. $1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90} = 1.082323234$

Use Prob. 14 in Sec. 11.1.

13. $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots = \frac{\pi^4}{96} = 1.014678032$

Use Prob. 17 in Sec. 11.1.

14. $\int_{-\pi}^{\pi} \cos^4 x \, dx = \frac{3\pi}{4}$

15. $\int_{-\pi}^{\pi} \cos^6 x \, dx = \frac{5\pi}{8}$

11.5 Sturm–Liouville Problems. Orthogonal Functions

The idea of the Fourier series was to represent general periodic functions in terms of cosines and sines. The latter formed a *trigonometric system*. This trigonometric system has the desirable property of orthogonality which allows us to compute the coefficient of the Fourier series by the Euler formulas.

The question then arises, can this approach be generalized? That is, can we replace the trigonometric system of Sec. 11.1 by other *orthogonal systems (sets of other orthogonal functions)*? The answer is “yes” and will lead to generalized Fourier series, including the Fourier–Legendre series and the Fourier–Bessel series in Sec. 11.6.

To prepare for this generalization, we first have to introduce the concept of a Sturm–Liouville Problem. (The motivation for this approach will become clear as you read on.) Consider a second-order ODE of the form

Example 3 confirms, from this new perspective, that the trigonometric system underlying the Fourier series is orthogonal, as we knew from Sec. 11.1.

EXAMPLE 4 Application of Theorem 1. Orthogonality of the Legendre Polynomials

Legendre’s equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ may be written

$$[(1 - x^2)y']' + \lambda y = 0 \quad \lambda = n(n + 1).$$

Hence, this is a Sturm–Liouville equation (1) with $p = 1 - x^2$, $q = 0$, and $r = 1$. Since $p(-1) = p(1) = 0$, we need no boundary conditions, but have a “singular” Sturm–Liouville problem on the interval $-1 \leq x \leq 1$. We know that for $n = 0, 1, \dots$, hence $\lambda = 0, 1 \cdot 2, 2 \cdot 3, \dots$, the Legendre polynomials $P_n(x)$ are solutions of the problem. Hence these are the eigenfunctions. From Theorem 1 it follows that they are orthogonal on that interval, that is,

$$(10) \quad \int_{-1}^1 P_m(x)P_n(x) dx = 0 \quad (m \neq n). \blacksquare$$

What we have seen is that the trigonometric system, underlying the Fourier series, is a solution to a Sturm–Liouville problem, as shown in Example 1, and that this trigonometric system is orthogonal, which we knew from Sec. 11.1 and confirmed in Example 3.

PROBLEM SET 11.5

- 1. Proof of Theorem 1.** Carry out the details in Cases 3 and 4.

2–6 ORTHOGONALITY

- 2. Normalization of eigenfunctions** y_m of (1), (2) means that we multiply y_m by a nonzero constant c_m such that $c_m y_m$ has norm 1. Show that $z_m = c_m y_m$ with any $c \neq 0$ is an eigenfunction for the eigenvalue corresponding to y_m .
- 3. Change of x .** Show that if the functions $y_0(x), y_1(x), \dots$ form an orthogonal set on an interval $a \leq x \leq b$ (with $r(x) = 1$), then the functions $y_0(ct + k), y_1(ct + k), \dots, c > 0$, form an orthogonal set on the interval $(a - k)/c \leq t \leq (b - k)/c$.
- 4. Change of x .** Using Prob. 3, derive the orthogonality of $1, \cos \pi x, \sin \pi x, \cos 2\pi x, \sin 2\pi x, \dots$ on $-1 \leq x \leq 1$ ($r(x) = 1$) from that of $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots$ on $-\pi \leq x \leq \pi$.
- 5. Legendre polynomials.** Show that the functions $P_n(\cos \theta), n = 0, 1, \dots$, from an orthogonal set on the interval $0 \leq \theta \leq \pi$ with respect to the weight function $\sin \theta$.
- 6. Transformation to Sturm–Liouville form.** Show that $y'' + fy' + (g + \lambda h)y = 0$ takes the form (1) if you

set $p = \exp(\int f dx), q = pg, r = hp$. Why would you do such a transformation?

7–15 STURM–LIOUVILLE PROBLEMS

Find the eigenvalues and eigenfunctions. Verify orthogonality. Start by writing the ODE in the form (1), using Prob. 6. Show details of your work.

- 7.** $y'' + \lambda y = 0, y(0) = 0, y(10) = 0$
- 8.** $y'' + \lambda y = 0, y(0) = 0, y(L) = 0$
- 9.** $y'' + \lambda y = 0, y(0) = 0, y'(L) = 0$
- 10.** $y'' + \lambda y = 0, y(0) = y(1), y'(0) = y'(1)$
- 11.** $(y'/x)' + (\lambda + 1)y/x^3 = 0, y(1) = 0, y(e^\pi) = 0.$ (Set $x = e^t$.)
- 12.** $y'' - 2y' + (\lambda + 1)y = 0, y(0) = 0, y(1) = 0$
- 13.** $y'' + 8y' + (\lambda + 16)y = 0, y(0) = 0, y(\pi) = 0$
- 14. TEAM PROJECT. Special Functions. Orthogonal polynomials** play a great role in applications. For this reason, Legendre polynomials and various other orthogonal polynomials have been studied extensively; see Refs. [GenRef1], [GenRef10] in App. 1. Consider some of the most important ones as follows.

Here we can let $k \rightarrow \infty$, because the left sides form a monotone increasing sequence that is bounded by the right side, so that we have convergence by the familiar Theorem 1 in App. A.3.3 Hence

$$(17) \quad \sum_{m=0}^{\infty} a_m^2 \leq \|f\|^2.$$

Furthermore, if y_0, y_1, \dots is complete in a set of functions S , then (13) holds for every f belonging to S . By (13) this implies equality in (16) with $k \rightarrow \infty$. Hence in the case of completeness every f in S satisfies the so-called **Parseval equality** (analog of (8) in Sec. 11.4)

$$(18) \quad \sum_{m=0}^{\infty} a_m^2 = \|f\|^2 = \int_a^b r(x)f(x)^2 dx.$$

As a consequence of (18) we prove that in the case of *completeness* there is no function orthogonal to every function of the orthonormal set, with the trivial exception of a function of zero norm:

THEOREM 2

Completeness

Let y_0, y_1, \dots be a complete orthonormal set on $a \leq x \leq b$ in a set of functions S . Then if a function f belongs to S and is orthogonal to every y_m , it must have norm zero. In particular, if f is continuous, then f must be identically zero.

PROOF

Since f is orthogonal to every y_m , the left side of (18) must be zero. If f is continuous, then $\|f\| = 0$ implies $f(x) \equiv 0$, as can be seen directly from (5) in Sec. 11.5 with f instead of y_m because $r(x) > 0$ by assumption. ■

PROBLEM SET 11.6

1-7 FOURIER-LEGENDRE SERIES

Showing the details, develop

1. $63x^5 - 90x^3 + 35x$
2. $(x + 1)^2$
3. $1 - x^4$
4. $1, x, x^2, x^3, x^4$
5. Prove that if $f(x)$ is even (is odd, respectively), its Fourier-Legendre series contains only $P_m(x)$ with even m (only $P_m(x)$ with odd m , respectively). Give examples.
6. What can you say about the coefficients of the Fourier-Legendre series of $f(x)$ if the Maclaurin series of $f(x)$ contains only powers x^{4m} ($m = 0, 1, 2, \dots$)?
7. What happens to the Fourier-Legendre series of a polynomial $f(x)$ if you change a coefficient of $f(x)$? Experiment. Try to prove your answer.

8-13 CAS EXPERIMENT

FOURIER-LEGENDRE SERIES. Find and graph (on common axes) the partial sums up to S_{m_0} whose graph practically coincides with that of $f(x)$ within graphical accuracy. State m_0 . On what does the size of m_0 seem to depend?

8. $f(x) = \sin \pi x$
9. $f(x) = \sin 2\pi x$
10. $f(x) = e^{-x^2}$
11. $f(x) = (1 + x^2)^{-1}$
12. $f(x) = J_0(\alpha_{0,1}x)$, $\alpha_{0,1}$ = the first positive zero of $J_0(x)$
13. $f(x) = J_0(\alpha_{0,2}x)$, $\alpha_{0,2}$ = the second positive zero of $J_0(x)$

- 14. TEAM PROJECT. Orthogonality on the Entire Real Axis. Hermite Polynomials.**⁸ These orthogonal polynomials are defined by $He_0(x) = 1$ and

$$(19) \quad He_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2/2}), \quad n = 1, 2, \dots$$

REMARK. As is true for many special functions, the literature contains more than one notation, and one sometimes defines as Hermite polynomials the functions

$$H_0^* = 1, \quad H_n^*(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

This differs from our definition, which is preferred in applications.

- (a) **Small Values of n .** Show that

$$\begin{aligned} He_1(x) &= x, & He_2(x) &= x^2 - 1, \\ He_3(x) &= x^3 - 3x, & He_4(x) &= x^4 - 6x^2 + 3. \end{aligned}$$

- (b) **Generating Function.** A generating function of the Hermite polynomials is

$$(20) \quad e^{tx - t^2/2} = \sum_{n=0}^{\infty} a_n(x) t^n$$

because $He_n(x) = n! a_n(x)$. Prove this. *Hint:* Use the formula for the coefficients of a Maclaurin series and note that $tx - \frac{1}{2}t^2 = \frac{1}{2}x^2 - \frac{1}{2}(x - t)^2$.

- (c) **Derivative.** Differentiating the generating function with respect to x , show that

$$(21) \quad He_n'(x) = nHe_{n-1}(x).$$

- (d) **Orthogonality on the x -Axis** needs a weight function that goes to zero sufficiently fast as $x \rightarrow \pm\infty$. (Why?)

Show that the Hermite polynomials are orthogonal on $-\infty < x < \infty$ with respect to the weight function $r(x) = e^{-x^2/2}$. *Hint.* Use integration by parts and (21).

- (e) **ODEs.** Show that

$$(22) \quad He_n'(x) = xHe_n(x) - He_{n+1}(x).$$

Using this with $n - 1$ instead of n and (21), show that $y = He_n(x)$ satisfies the ODE

$$(23) \quad y'' = xy' + ny = 0.$$

Show that $w = e^{-x^2/4}y$ is a solution of **Weber's equation**

$$(24) \quad w'' + (n + \frac{1}{2} - \frac{1}{4}x^2)w = 0 \quad (n = 0, 1, \dots).$$

- 15. CAS EXPERIMENT. Fourier-Bessel Series.** Use Example 2 and $R = 1$, so that you get the series

$$(25) \quad f(x) = a_1 J_0(\alpha_{0,1}x) + a_2 J_0(\alpha_{0,2}x) + a_3 J_0(\alpha_{0,3}x) + \dots$$

With the zeros $\alpha_{0,1}, \alpha_{0,2}, \dots$ from your CAS (see also Table A1 in App. 5).

- (a) Graph the terms $J_0(\alpha_{0,1}x), \dots, J_0(\alpha_{0,10}x)$ for $0 \leq x \leq 1$ on common axes.

(b) Write a program for calculating partial sums of (25). Find out for what $f(x)$ your CAS can evaluate the integrals. Take two such $f(x)$ and comment empirically on the speed of convergence by observing the decrease of the coefficients.

- (c) Take $f(x) = 1$ in (25) and evaluate the integrals for the coefficients analytically by (21a), Sec. 5.4, with $\nu = 1$. Graph the first few partial sums on common axes.

11.7 Fourier Integral

Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only. Sections 11.2 and 11.3 first illustrated this, and various further applications follow in Chap. 12. Since, of course, many problems involve functions that are *nonperiodic and are of interest on the whole x -axis*, we ask what can be done to extend the method of Fourier series to such functions. This idea will lead to "Fourier integrals."

In Example 1 we start from a special function f_L of period $2L$ and see what happens to its Fourier series if we let $L \rightarrow \infty$. Then we do the same for an *arbitrary* function f_L of period $2L$. This will motivate and suggest the main result of this section, which is an integral representation given in Theorem 1 below.

⁸CHARLES HERMITE (1822–1901), French mathematician, is known for his work in algebra and number theory. The great HENRI POINCARÉ (1854–1912) was one of his students.

Similarly,

$$\begin{aligned} a_{n-4} &= -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} \\ &= \frac{(2n-4)!}{2^{n2}!(n-2)!(n-4)!} \end{aligned}$$

and so on, and in general, when $n - 2m \geq 0$,

$$(10) \quad a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^{nm}!(n-m)!(n-2m)!}.$$

The resulting solution of Legendre's differential equation (1) is called the **Legendre polynomial of degree n** and is denoted by $P_n(x)$.

From (10) we obtain

$$(11) \quad \begin{aligned} P_n(x) &= \sum_{m=0}^M (-1)^m \frac{(2n-2m)!}{2^{nm}!(n-m)!(n-2m)!} x^{n-2m} \\ &= \frac{(2n)!}{2^n(n!)^2} x^n - \frac{(2n-2)!}{2^{n-1}!(n-1)!(n-2)!} x^{n-2} + \dots \end{aligned}$$

where $M = n/2$ or $(n-1)/2$, whichever is an integer. The first few of these functions are (Fig. 107)

$$(11') \quad \begin{aligned} P_0(x) &= 1, & P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1), & P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3), & P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) \end{aligned}$$

and so on. You may now program (11) on your CAS and calculate $P_n(x)$ as needed.

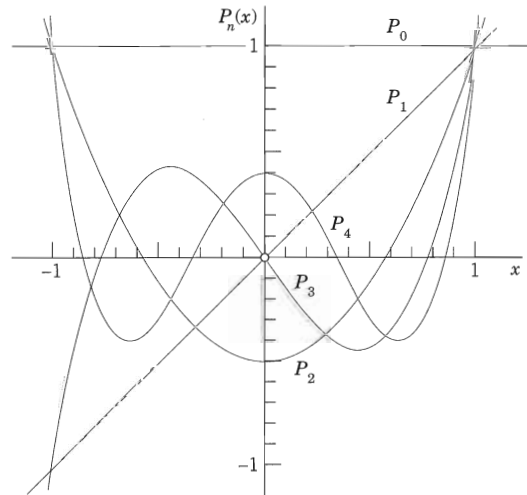


Fig. 107. Legendre polynomials

PROBLEM SET 11.7

1-6 EVALUATION OF INTEGRALS

Show that the integral represents the indicated function.
Hint. Use (5), (10), or (11); the integral tells you which one, and its value tells you what function to consider. Show your work in detail.

$$1. \int_0^{\infty} \frac{\cos xw + w \sin xw}{1 + w^2} dx = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

$$2. \int_0^{\infty} \frac{\sin \pi w \sin xw}{1 - w^2} dw = \begin{cases} \frac{\pi}{2} \sin x & \text{if } 0 \leq x \leq \pi \\ 0 & \text{if } x > \pi \end{cases}$$

$$3. \int_0^{\infty} \frac{1 - \cos \pi w}{w} \sin xw dw = \begin{cases} \frac{1}{2}\pi & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$

$$4. \int_0^{\infty} \frac{\cos \frac{1}{2} \pi w}{1 - w^2} \cos xw dw = \begin{cases} \frac{1}{2}\pi \cos x & \text{if } 0 < |x| < \frac{1}{2}\pi \\ 0 & \text{if } |x| \geq \frac{1}{2}\pi \end{cases}$$

$$5. \int_0^{\infty} \frac{\sin w - w \cos w}{w^2} \sin xw dw = \begin{cases} \frac{1}{2}\pi x & \text{if } 0 < x < 1 \\ \frac{1}{4}\pi & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$6. \int_0^{\infty} \frac{w^3 \sin xw}{w^4 + 4} dw = \frac{1}{2}\pi e^{-x} \cos x \quad \text{if } x > 0$$

7-12 FOURIER COSINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (10).

$$7. f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$8. f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$9. f(x) = 1/(1 + x^2) \quad [x > 0. \text{ Hint. See (13).}]$$

$$10. f(x) = \begin{cases} a^2 - x^2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

$$11. f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$

$$12. f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

13. CAS EXPERIMENT. Approximate Fourier Cosine Integrals. Graph the integrals in Prob. 7, 9, and 11 as

functions of x . Graph approximations obtained by replacing ∞ with finite upper limits of your choice. Compare the quality of the approximations. Write a short report on your empirical results and observations.

14. PROJECT. Properties of Fourier Integrals

(a) **Fourier cosine integral.** Show that (10) implies

$$(a1) \quad f(ax) = \frac{1}{a} \int_0^{\infty} A\left(\frac{w}{a}\right) \cos xw dw$$

($a > 0$) (Scale change)

$$(a2) \quad xf(x) = \int_0^{\infty} B^*(w) \sin xw dw,$$

$$B^* = -\frac{dA}{dw}, \quad A \text{ as in (10)}$$

$$(a3) \quad x^2 f(x) = \int_0^{\infty} A^*(w) \cos xw dw,$$

$$A^* = -\frac{d^2 A}{dw^2}.$$

(b) Solve Prob. 8 by applying (a3) to the result of Prob. 7.

(c) Verify (a2) for $f(x) = 1$ if $0 < x < a$ and $f(x) = 0$ if $x > a$.

(d) **Fourier sine integral.** Find formulas for the Fourier sine integral similar to those in (a).

15. CAS EXPERIMENT. Sine Integral. Plot $\text{Si}(u)$ for positive u . Does the sequence of the maximum and minimum values give the impression that it converges and has the limit $\pi/2$? Investigate the Gibbs phenomenon graphically.

16-20 FOURIER SINE INTEGRAL REPRESENTATIONS

Represent $f(x)$ as an integral (11).

$$16. f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

$$17. f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$18. f(x) = \begin{cases} \cos x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$

$$19. f(x) = \begin{cases} e^x & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$20. f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

PROBLEM SET 11.8

1-8 FOURIER COSINE TRANSFORM

- Find the cosine transform $\hat{f}_c(w)$ of $f(x) = 1$ if $0 < x < 1$, $f(x) = -1$ if $1 < x < 2$, $f(x) = 0$ if $x > 2$.
- Find f in Prob. 1 from the answer \hat{f}_c .
- Find $\hat{f}_c(w)$ for $f(x) = x$ if $0 < x < 2$, $f(x) = 0$ if $x > 2$.
- Derive formula 3 in Table I of Sec. 11.10 by integration.
- Find $\hat{f}_c(w)$ for $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.
- Continuity assumptions.** Find $\hat{g}_c(w)$ for $g(x) = 2$ if $0 < x < 1$, $g(x) = 0$ if $x > 1$. Try to obtain from it $\hat{f}_c(w)$ for $f(x)$ in Prob. 5 by using (5a).
- Existence?** Does the Fourier cosine transform of $x^{-1} \sin x$ ($0 < x < \infty$) exist? Of $x^{-1} \cos x$? Give reasons.
- Existence?** Does the Fourier cosine transform of $f(x) = k = \text{const}$ ($0 < x < \infty$) exist? The Fourier sine transform?

9-15 FOURIER SINE TRANSFORM

- Find $\mathcal{F}_s(e^{-ax})$, $a > 0$, by integration.
- Obtain the answer to Prob. 9 from (5b).
- Find $f_s(w)$ for $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.
- Find $\mathcal{F}_s(xe^{-x^2/2})$ from (4b) and a suitable formula in Table I of Sec. 11.10.
- Find $\mathcal{F}_s(e^{-x})$ from (4a) and formula 3 of Table I in Sec. 11.10.
- Gamma function.** Using formulas 2 and 4 in Table II of Sec. 11.10, prove $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [(30) in App. A3.1], a value needed for Bessel functions and other applications.
- WRITING PROJECT. Finding Fourier Cosine and Sine Transforms.** Write a short report on ways of obtaining these transforms, with illustrations by examples of your own.

11.9 Fourier Transform. Discrete and Fast Fourier Transforms

In Sec. 11.8 we derived two real transforms. Now we want to derive a complex transform that is called the **Fourier transform**. It will be obtained from the complex Fourier integral, which will be discussed next.

Complex Form of the Fourier Integral

The (real) Fourier integral is [see (4), (5), Sec. 11.7]

$$f(x) = \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx] dw$$

where

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv, \quad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv dv.$$

Substituting A and B into the integral for f , we have

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(v) [\cos wv \cos wx + \sin wv \sin wx] dv dw.$$

PROBLEM SET 11.9

1. **Review in complex.** Show that $1/i = -i$, $e^{-ix} = \cos x - i \sin x$, $e^{ix} + e^{-ix} = 2 \cos x$, $e^{ix} - e^{-ix} = 2i \sin x$, $e^{ikx} = \cos kx + i \sin kx$.

2-11 FOURIER TRANSFORMS BY INTEGRATION

Find the Fourier transform of $f(x)$ (without using Table III in Sec. 11.10). Show details.

$$2. f(x) = \begin{cases} e^{2ix} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$3. f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$4. f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \end{cases}$$

$$5. f(x) = \begin{cases} e^x & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

$$6. f(x) = e^{-|x|} \quad (-\infty < x < \infty)$$

$$7. f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

$$8. f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$9. f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$10. f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$11. f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

12-17 USE OF TABLE III IN SEC. 11.10.
OTHER METHODS

12. Find $\mathcal{F}\{f(x)\}$ for $f(x) = xe^{-x}$ if $x > 0$, $f(x) = 0$ if $x < 0$, by (9) in the text and formula 5 in Table III (with $a = 1$). *Hint.* Consider xe^{-x} and e^{-x} .
13. Obtain $\mathcal{F}\{e^{-x^2/2}\}$ from Table III.
14. In Table III obtain formula 7 from formula 8.
15. In Table III obtain formula 1 from formula 2.
16. **TEAM PROJECT. Shifting** (a) Show that if $f(x)$ has a Fourier transform, so does $f(x - a)$, and $\mathcal{F}\{f(x - a)\} = e^{-iwa} \mathcal{F}\{f(x)\}$.
 (b) Using (a), obtain formula 1 in Table III, Sec. 11.10, from formula 2.
 (c) **Shifting on the w -Axis.** Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w - a)$ is the Fourier transform of $e^{iax}f(x)$.
 (d) Using (c), obtain formula 7 in Table III from 1 and formula 8 from 2.
17. What could give you the idea to solve Prob. 11 by using the solution of Prob. 9 and formula (9) in the text? Would this work?

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18. Verify the calculations in Example 4 of the text.
19. Find the transform of a general signal $f = [f_1 \ f_2 \ f_3 \ f_4]^T$ of four values.
20. Find the inverse matrix in Example 4 of the text and use it to recover the given signal.
21. Find the transform (the frequency spectrum) of a general signal of two values $[f_1 \ f_2]^T$.
22. Recreate the given signal in Prob. 21 from the frequency spectrum obtained.
23. Show that for a signal of eight sample values, $w = e^{-i/4} = (1 - i)/\sqrt{2}$. Check by squaring.
24. Write the Fourier matrix F for a sample of eight values explicitly.
25. **CAS Problem.** Calculate the inverse of the 8×8 Fourier matrix. Transform a general sample of eight values and transform it back to the given data.