THEOREM 3

Existence Theorem for Laplace Transforms

If $f(t)$ is defined and piecewise continuous on every finite interval on the semi-axis $t \geqq 0$ and satisfies (2) *for all $t \geqq 0$ and some constants M and k, then the Laplace transform $\mathcal{L}(f)$ exists for all $s > k$.*

PROOF Since $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any finite interval on the t-axis. From (2), assuming that $s > k$ (to be needed for the existence of the last of the following integrals), we obtain the proof of the existence of $\mathcal{L}(f)$ from

$$|\mathcal{L}(f)| = \left|\int_0^{\infty} e^{-st}f(t)\,dt\right| \leqq \int_0^{\infty} |f(t)|e^{-st}\,dt \leqq \int_0^{\infty} Me^{kt}e^{-st}\,dt = \frac{M}{s-k}.$$ ■

Note that (2) can be readily checked. For instance, $\cosh t < e^t$, $t^n < n!e^t$ (because $t^n/n!$ is a single term of the Maclaurin series), and so on. A function that does not satisfy (2) for any M and k is e^{t^2} (take logarithms to see it). We mention that the conditions in Theorem 3 are sufficient rather than necessary (see Prob. 22).

Uniqueness. If the Laplace transform of a given function exists, it is uniquely determined. Conversely, it can be shown that if two functions (both defined on the positive real axis) have the same transform, these functions cannot differ over an interval of positive length, although they may differ at isolated points (see Ref. [A14] in App. 1). Hence we may say that the inverse of a given transform is essentially unique. In particular, if two *continuous* functions have the same transform, they are completely identical.

PROBLEM SET 6.1

1–16 LAPLACE TRANSFORMS

Find the transform. Show the details of your work. Assume that a, b, ω, θ are constants.

1. $3t + 12$

2. $(a - bt)^2$

3. $\cos \pi t$

4. $\cos^2 \omega t$

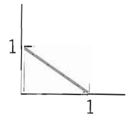
5. $e^{2t} \sinh t$

6. $e^{-t} \sinh 4t$

7. $\sin (\omega t + \theta)$

8. $1.5 \sin (3t - \pi/2)$

9.



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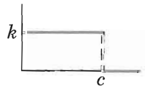

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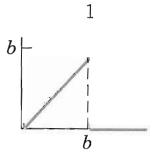

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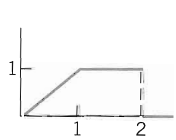

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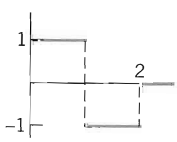

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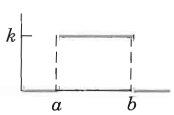

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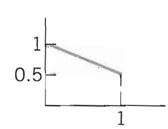

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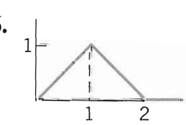

17–24 SOME THEORY

17. Table 6.1. Convert this table to a table for finding inverse transforms (with obvious changes, e.g., $\mathcal{L}^{-1}(1/s^n) = t^{n-1}/(n-1)$, etc).

18. Using $\mathcal{L}(f)$ in Prob. 10, find $\mathcal{L}(f_1)$, where $f_1(t) = 0$ if $t \leqq 2$ and $f_1(t) = 1$ if $t > 2$.

19. Table 6.1. Derive formula 6 from formulas 9 and 10.

20. Nonexistence. Show that e^{t^2} does not satisfy a condition of the form (2).

21. Nonexistence. Give simple examples of functions (defined for all $t \geqq 0$) that have no Laplace transform.

22. Existence. Show that $\mathcal{L}(1/\sqrt{t}) = \sqrt{\pi/s}$. [Use (30) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ in App. 3.1.] Conclude from this that the conditions in Theorem 3 are sufficient but not necessary for the existence of a Laplace transform.

23. Change of scale. If $\mathcal{L}(f(t)) = F(s)$ and c is any positive constant, show that $\mathcal{L}(f(ct)) = F(s/c)/c$ (***Hint:*** Use (1).) Use this to obtain $\mathcal{L}(\cos \omega t)$ from $\mathcal{L}(\cos t)$.

24. Inverse transform. Prove that $\mathcal{L}^{-1}$ is linear. *Hint:* Use the fact that $\mathcal{L}$ is linear.

25–32 INVERSE LAPLACE TRANSFORMS

Given $F(s) = \mathcal{L}(f)$, find $f(t)$. a, b, L, n are constants. Show the details of your work.

25. $\dfrac{0.2s + 1.8}{s^2 + 3.24}$

26. $\dfrac{5s + 1}{s^2 - 25}$

27. $\dfrac{s}{L^2s^2 + n^2\pi^2}$

28. $\dfrac{1}{(s + \sqrt{2})(s - \sqrt{3})}$

29. $\dfrac{12}{s^4} - \dfrac{228}{s^6}$

30. $\dfrac{4s + 32}{s^2 - 16}$

31. $\dfrac{s + 10}{s^2 - s - 2}$

32. $\dfrac{1}{(s + a)(s + b)}$

33–45 APPLICATION OF s-SHIFTING

In Probs. 33–36 find the transform. In Probs. 37–45 find the inverse transform. Show the details of your work.

33. t^2e^{-3t}

34. $ke^{-at} \cos \omega t$

35. $0.5e^{-4.5t} \sin 2\pi t$

36. $\sinh t \cos t$

37. $\dfrac{\pi}{(s + \pi)^2}$

38. $\dfrac{6}{(s + 1)^3}$

39. $\dfrac{21}{(s + \sqrt{2})^4}$

40. $\dfrac{4}{s^2 - 2s - 3}$

41. $\dfrac{\pi}{s^2 + 10\pi s + 24\pi^2}$

42. $\dfrac{a_0}{s + 1} + \dfrac{a_1}{(s + 1)^2} + \dfrac{a_2}{(s + 1)^3}$

43. $\dfrac{2s - 1}{s^2 - 6s + 18}$

44. $\dfrac{a(s + k) + b\pi}{(s + k)^2 + \pi^2}$

45. $\dfrac{k_0(s + a) + k_1}{(s + a)^2}$

6.2 Transforms of Derivatives and Integrals. ODEs

The Laplace transform is a method of solving ODEs and initial value problems. The crucial idea is that ***operations of calculus on functions are replaced by operations of algebra on transforms***. Roughly, *differentiation* of $f(t)$ will correspond to *multiplication* of $\mathcal{L}(f)$ by s (see Theorems 1 and 2) and *integration* of $f(t)$ to *division* of $\mathcal{L}(f)$ by s. To solve ODEs, we must first consider the Laplace transform of derivatives. You have encountered such an idea in your study of logarithms. Under the application of the natural logarithm, a product of numbers becomes a sum of their logarithms, a division of numbers becomes their difference of logarithms (see Appendix 3, formulas (2), (3)). To simplify calculations was one of the main reasons that logarithms were invented in pre-computer times.

THEOREM 1

Laplace Transform of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) \tag{1}$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0). \tag{2}$$

Formula (1) *holds if $f(t)$ is continuous for all $t \geqq 0$ and satisfies the growth restriction* (2) *in Sec.* 6.1 *and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Similarly,* (2) *holds if f and f' are continuous for all $t \geqq 0$ and satisfy the growth restriction and f'' is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$.*

EXAMPLE 6 **Shifted Data Problems**

This means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$. For such a problem set $t = \tilde{t} + t_0$, so that $t = t_0$ gives $\tilde{t} = 0$ and the Laplace transform can be applied. For instance, solve

$$y'' + y = 2t, \quad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \quad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}.$$

Solution. We have $t_0 = \frac{1}{4}\pi$ and we set $t = \tilde{t} + \frac{1}{4}\pi$. Then the problem is

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{1}{4}\pi), \quad \tilde{y}(0) = \tfrac{1}{2}\pi, \quad \tilde{y}'(0) = 2 - \sqrt{2}$$

where $\tilde{y}(\tilde{t}) = y(t)$. Using (2) and Table 6.1 and denoting the transform of $\tilde{y}$ by $\tilde{Y}$, we see that the subsidiary equation of the "shifted" initial value problem is

$$s^2\tilde{Y} - s \cdot \tfrac{1}{2}\pi - (2 - \sqrt{2}) + \tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s}, \quad \text{thus} \quad (s^2 + 1)\tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}.$$

Solving this algebraically for $\tilde{Y}$, we obtain

$$\tilde{Y} = \frac{2}{(s^2 + 1)s^2} + \frac{\frac{1}{2}\pi}{(s^2 + 1)s} + \frac{\frac{1}{2}\pi s}{s^2 + 1} + \frac{2 - \sqrt{2}}{s^2 + 1}.$$

The inverse of the first two terms can be seen from Example 3 (with $\omega = 1$), and the last two terms give cos and sin,

$$\begin{aligned} \tilde{y} = \mathcal{L}^{-1}(\tilde{Y}) &= 2(\tilde{t} - \sin \tilde{t}) + \tfrac{1}{2}\pi(1 - \cos \tilde{t}) + \tfrac{1}{2}\pi \cos \tilde{t} + (2 - \sqrt{2}) \sin \tilde{t} \\ &= 2\tilde{t} + \tfrac{1}{2}\pi - \sqrt{2} \sin \tilde{t}. \end{aligned}$$

Now $\tilde{t} = t - \frac{1}{4}\pi$, $\sin \tilde{t} = \frac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer (the solution) is

$$y = 2t - \sin t + \cos t.$$

■

PROBLEM SET 6.2

1–11 INITIAL VALUE PROBLEMS (IVPS)

Solve the IVPs by the Laplace transform. If necessary, use partial fraction expansion as in Example 4 of the text. Show all details.

1. $y' + 5.2y = 19.4 \sin 2t, \quad y(0) = 0$

2. $y' + 2y = 0, \quad y(0) = 1.5$

3. $y'' - y' - 6y = 0, \quad y(0) = 11, \quad y'(0) = 28$

4. $y'' + 9y = 10e^{-t}, \quad y(0) = 0, \quad y'(0) = 0$

5. $y'' - \frac{1}{4}y = 0, \quad y(0) = 12, \quad y'(0) = 0$

6. $y'' - 6y' + 5y = 29 \cos 2t, \quad y(0) = 3.2,$ $y'(0) = 6.2$

7. $y'' + 7y' + 12y = 21e^{3t}, \quad y(0) = 3.5,$ $y'(0) = -10$

8. $y'' - 4y' + 4y = 0, \quad y(0) = 8.1, \quad y'(0) = 3.9$

9. $y'' - 4y' + 3y = 6t - 8, \quad y(0) = 0, \quad y'(0) = 0$

10. $y'' + 0.04y = 0.02t^2, \quad y(0) = -25, \quad y'(0) = 0$

11. $y'' + 3y' + 2.25y = 9t^3 + 64, \quad y(0) = 1,$ $y'(0) = 31.5$

12–15 SHIFTED DATA PROBLEMS

Solve the shifted data IVPs by the Laplace transform. Show the details.

12. $y'' - 2y' - 3y = 0, \quad y(4) = -3,$ $y'(4) = -17$

13. $y' - 6y = 0, \quad y(-1) = 4$

14. $y'' + 2y' + 5y = 50t - 100, \quad y(2) = -4,$ $y'(2) = 14$

15. $y'' + 3y' - 4y = 6e^{2t-3}, \quad y(1.5) = 4,$ $y'(1.5) = 5$

16–21 OBTAINING TRANSFORMS BY DIFFERENTIATION

Using (1) or (2), find $\mathcal{L}(f)$ if $f(t)$ equals:

16. $t \cos 4t$

17. te^{-at}

18. $\cos^2 2t$

19. $\sin^2 \omega t$

20. $\sin^4 t$. Use Prob. 19.

21. $\cosh^2 t$

22. PROJECT. Further Results by Differentiation. Proceeding as in Example 1, obtain

$$\textbf{(a)}\quad \mathcal{L}(t\cos\omega t)=\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

and from this and Example 1: **(b)** formula 21, **(c)** 22, **(d)** 23 in Sec. 6.9,

$$\textbf{(e)}\quad \mathcal{L}(t\cosh at)=\frac{s^2+a^2}{(s^2-a^2)^2},$$

$$\textbf{(f)}\quad \mathcal{L}(t\sinh at)=\frac{2as}{(s^2-a^2)^2}.$$

23–29 INVERSE TRANSFORMS BY INTEGRATION

Using Theorem 3, find $f(t)$ if $\mathcal{L}(F)$ equals:

23. $\dfrac{3}{s^2+s/4}$ **24.** $\dfrac{20}{s^3-2\pi s^2}$

25. $\dfrac{1}{s(s^2+\omega^2)}$ **26.** $\dfrac{1}{s^4-s^2}$

27. $\dfrac{s+1}{s^4+9s^2}$ **28.** $\dfrac{3s+4}{s^4+k^2s^2}$

29. $\dfrac{1}{s^3+as^2}$

30. PROJECT. Comments on Sec. 6.2. **(a)** Give reasons why Theorems 1 and 2 are more important than Theorem 3.

(b) Extend Theorem 1 by showing that if $f(t)$ is continuous, except for an ordinary discontinuity (finite jump) at some $t = a\ (>0)$, the other conditions remaining as in Theorem 1, then (see Fig. 117)

$$(1^*)\quad \mathcal{L}(f')=s\mathcal{L}(f)-f(0)-[f(a+0)-f(a-0)]e^{-as}.$$

(c) Verify (1*) for $f(t)=e^{-t}$ if $0<t<1$ and 0 if $t>1$.

(d) Compare the Laplace transform of solving ODEs with the method in Chap. 2. Give examples of your own to illustrate the advantages of the present method (to the extent we have seen them so far).

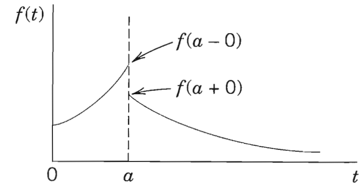

Fig. 117. Formula (1*)

6.3 Unit Step Function (Heaviside Function). Second Shifting Theorem (t-Shifting)

This section and the next one are extremely important because we shall now reach the point where the Laplace transform method shows its real power in applications and its superiority over the classical approach of Chap. 2. The reason is that we shall introduce two auxiliary functions, the *unit step function* or *Heaviside function* $u(t-a)$ (below) and *Dirac's delta* $\delta(t-a)$ (in Sec. 6.4). These functions are suitable for solving ODEs with complicated right sides of considerable engineering interest, such as single waves, inputs (driving forces) that are discontinuous or act for some time only, periodic inputs more general than just cosine and sine, or impulsive forces acting for an instant (hammerblows, for example).

Unit Step Function (Heaviside Function) $u(t-a)$

The **unit step function** or **Heaviside function** $u(t-a)$ is 0 for $t<a$, has a jump of size 1 at $t=a$ (where we can leave it undefined), and is 1 for $t>a$, in a formula:

$$(1)\qquad u(t-a)=\begin{cases}0 & \text{if } t<a\\ 1 & \text{if } t>a\end{cases}\qquad (a\geqq 0).$$

We set $s = -10$ and -100 and then equate the sums of the s^3 and s^2 terms to zero, obtaining (all values rounded)

$$\begin{array}{llr} (s = -10) & -4{,}000{,}000 = 90(10^2 + 400^2)A, & A = -0.27760 \\ (s = -100) & -40{,}000{,}000 = -90(100^2 + 400^2)B, & B = 2.6144 \\ (s^3\text{-terms}) & 0 = A + B + D, & D = -2.3368 \\ (s^2\text{-terms}) & 0 = 100A + 10B + 110D + K, & K = 258.66. \end{array}$$

Since $K = 258.66 = 0.6467 \cdot 400$, we thus obtain for the first term I_1 in $I = I_1 - I_2$

$$I_1 = -\frac{0.2776}{s+10} + \frac{2.6144}{s+100} - \frac{2.3368s}{s^2+400^2} + \frac{0.6467 \cdot 400}{s^2+400^2}.$$

From Table 6.1 in Sec. 6.1 we see that its inverse is

$$i_1(t) = -0.2776e^{-10t} + 2.6144e^{-100t} - 2.3368 \cos 400t + 0.6467 \sin 400t.$$

This is the current $i(t)$ when $0 < t < 2\pi$. It agrees for $0 < t < 2\pi$ with that in Example 1 of Sec. 2.9 (except for notation), which concerned the same *RLC*-circuit. Its graph in Fig. 63 in Sec. 2.9 shows that the exponential terms decrease very rapidly. Note that the present amount of work was substantially less.

The second term I_1 of I differs from the first term by the factor $e^{-2\pi s}$. Since $\cos 400(t - 2\pi) = \cos 400t$ and $\sin 400(t - 2\pi) = \sin 400t$, the second shifting theorem (Theorem 1) gives the inverse $i_2(t) = 0$ if $0 < t < 2\pi$, and for $> 2\pi$ it gives

$$i_2(t) = -0.2776e^{-10(t-2\pi)} + 2.6144e^{-100(t-2\pi)} - 2.3368 \cos 400t + 0.6467 \sin 400t.$$

Hence in $i(t)$ the cosine and sine terms cancel, and the current for $t > 2\pi$ is

$$i(t) = -0.2776(e^{-10t} - e^{-10(t-2\pi)}) + 2.6144(e^{-100t} - e^{-100(t-2\pi)}).$$

It goes to zero very rapidly, practically within 0.5 sec. ■

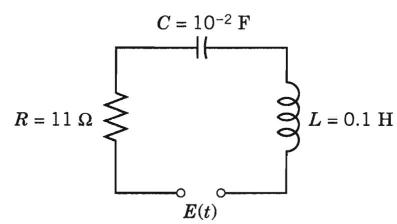

Fig. 125. *RLC*-circuit in Example 4

PROBLEM SET 6.3

1. **Report on Shifting Theorems.** Explain and compare the different roles of the two shifting theorems, using your own formulations and simple examples. Give no proofs.

2–11 SECOND SHIFTING THEOREM, UNIT STEP FUNCTION

Sketch or graph the given function, which is assumed to be zero outside the given interval. Represent it, using unit step functions. Find its transform. Show the details of your work.

2. $t\ (0 < t < 2)$
3. $t - 2\ (t > 2)$
4. $\cos 4t\ (0 < t < \pi)$
5. $e^t\ (0 < t < \pi/2)$
6. $\sin \pi t\ (2 < t < 4)$
7. $e^{-\pi t}\ (2 < t < 4)$
8. $t^2\ (1 < t < 2)$
9. $t^2\ (t > \frac{3}{2})$
10. $\sinh t\ (0 < t < 2)$
11. $\sin t\ (\pi/2 < t < \pi)$

12–17 INVERSE TRANSFORMS BY THE 2ND SHIFTING THEOREM

Find and sketch or graph $f(t)$ if $\mathcal{L}(f)$ equals

12. $e^{-3s}/(s-1)^3$
13. $6(1 - e^{-\pi s})/(s^2 + 9)$
14. $4(e^{-2s} - 2e^{-5s})/s$
15. e^{-3s}/s^4
16. $2(e^{-s} - e^{-3s})/(s^2 - 4)$
17. $(1 + e^{-2\pi(s+1)})(s+1)/((s+1)^2 + 1)$

18–27 IVPs, SOME WITH DISCONTINUOUS INPUT

Using the Laplace transform and showing the details, solve

18. $9y'' - 6y' + y = 0, \quad y(0) = 3, y'(0) = 1$

19. $y'' + 6y' + 8y = e^{-3t} - e^{-5t}, \quad y(0) = 0, y'(0) = 0$

20. $y'' + 10y' + 24y = 144t^2, \quad y(0) = 19/12,$ $y'(0) = -5$

21. $y'' + 9y = 8 \sin t$ if $0 < t < \pi$ and 0 if $t > \pi$; $y(0) = 0, y'(0) = 4$

22. $y'' + 3y' + 2y = 4t$ if $0 < t < 1$ and 8 if $t > 1$; $y(0) = 0, y'(0) = 0$

23. $y'' + y' - 2y = 3 \sin t - \cos t$ if $0 < t < 2\pi$ and $3 \sin 2t - \cos 2t$ if $t > 2\pi$; $\quad y(0) = 1, y'(0) = 0$

24. $y'' + 3y' + 2y = 1$ if $0 < t < 1$ and 0 if $t > 1$; $y(0) = 0, y'(0) = 0$

25. $y'' + y = t$ if $0 < t < 1$ and 0 if $t > 1$; $\quad y(0) = 0,$ $y'(0) = 0$

26. Shifted data. $y'' + 2y' + 5y = 10 \sin t$ if $0 < t < 2\pi$ and 0 if $t > 2\pi$; $\quad y(\pi) = 1, y'(\pi) = 2e^{-\pi} - 2$

27. Shifted data. $y'' + 4y = 8t^2$ if $0 < t < 5$ and 0 if $t > 5$; $\quad y(1) = 1 + \cos 2, y'(1) = 4 - 2 \sin 2$

28–40 MODELS OF ELECTRIC CIRCUITS

28–30 RL-CIRCUIT

Using the Laplace transform and showing the details, find the current $i(t)$ in the circuit in Fig. 126, assuming $i(0) = 0$ and:

28. $R = 1\ \text{k}\Omega\ (=1000\ \Omega), L = 1\ \text{H}, v = 0$ if $0 < t < \pi$, and $40 \sin t$ V if $t > \pi$

29. $R = 25\ \Omega, L = 0.1\ \text{H}, v = 490\ e^{-5t}$ V if $0 < t < 1$ and 0 if $t > 1$

30. $R = 10\ \Omega, L = 0.5\ \text{H}, v = 200t$ V if $0 < t < 2$ and 0 if $t > 2$

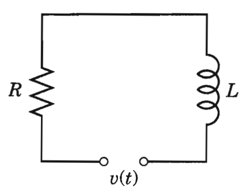

Fig. 126. Problems 28–30

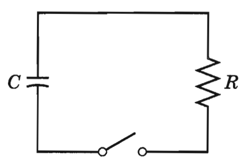

Fig. 127. Problem 31

31. Discharge in *RC*-circuit. Using the Laplace transform, find the charge $q(t)$ on the capacitor of capacitance C in Fig. 127 if the capacitor is charged so that its potential is V_0 and the switch is closed at $t = 0$.

32–34 RC-CIRCUIT

Using the Laplace transform and showing the details, find the current $i(t)$ in the circuit in Fig. 128 with $R = 10\ \Omega$ and $C = 10^{-2}$ F, where the current at $t = 0$ is assumed to be zero, and:

32. $v = 0$ if $t < 4$ and $14 \cdot 10^6 e^{-3t}$ V if $t > 4$

33. $v = 0$ if $t < 2$ and $100(t - 2)$ V if $t > 2$

34. $v(t) = 100$ V if $0.5 < t < 0.6$ and 0 otherwise. Why does $i(t)$ have jumps?

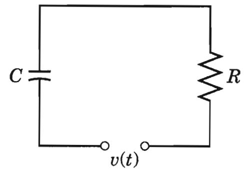

Fig. 128. Problems 32–34

35–37 LC-CIRCUIT

Using the Laplace transform and showing the details, find the current $i(t)$ in the circuit in Fig. 129, assuming zero initial current and charge on the capacitor and:

35. $L = 1\ \text{H}, C = 10^{-2}\ \text{F}, v = -9900 \cos t$ V if $\pi < t < 3\pi$ and 0 otherwise

36. $L = 1\ \text{H}, C = 0.25\ \text{F}, v = 200\ (t - \frac{1}{3}t^3)$ V if $0 < t < 1$ and 0 if $t > 1$

37. $L = 0.5\ \text{H}, C = 0.05\ \text{F}, v = 78 \sin t$ V if $0 < t < \pi$ and 0 if $t > \pi$

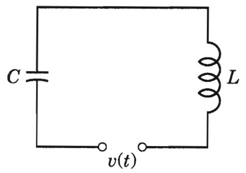

Fig. 129. Problems 35–37

38–40 RLC-CIRCUIT

Using the Laplace transform and showing the details, find the current $i(t)$ in the circuit in Fig. 130, assuming zero initial current and charge and:

38. $R = 4\ \Omega, L = 1\ \text{H}, C = 0.05\ \text{F}, v = 34e^{-t}$ V if $0 < t < 4$ and 0 if $t > 4$

39. $R = 2\ \Omega$, $L = 1$ H, $C = 0.5$ F, $v(t) = 1$ kV if $0 < t < 2$ and 0 if $t > 2$

40. $R = 2\ \Omega$, $L = 1$ H, $C = 0.1$ F, $v = 255 \sin t$ V if $0 < t < 2\pi$ and 0 if $t > 2\pi$

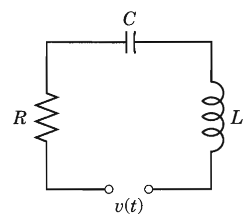

Fig. 130. Problems 38–40

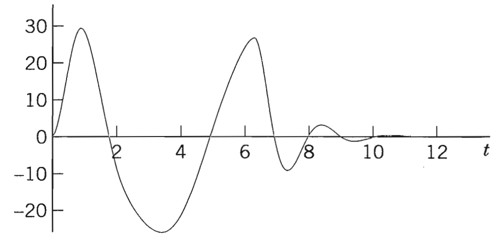

Fig. 131. Current in Problem 40

6.4 Short Impulses. Dirac's Delta Function. Partial Fractions

An airplane making a "hard" landing, a mechanical system being hit by a hammerblow, a ship being hit by a single high wave, a tennis ball being hit by a racket, and many other similar examples appear in everyday life. They are phenomena of an impulsive nature where actions of forces—mechanical, electrical, etc.—are applied over short intervals of time.

We can model such phenomena and problems by "Dirac's delta function," and solve them very effecively by the Laplace transform.

To model situations of that type, we consider the function

$$f_k(t - a) = \begin{cases} 1/k & \text{if } a \leqq t \leqq a + k \\ 0 & \text{otherwise} \end{cases} \qquad (1) \quad \text{(Fig. 132)}$$

(and later its limit as $k \to 0$). This function represents, for instance, a force of magnitude $1/k$ acting from $t = a$ to $t = a + k$, where k is positive and small. In mechanics, the integral of a force acting over a time interval $a \leqq t \leqq a + k$ is called the **impulse** of the force; similarly for electromotive forces $E(t)$ acting on circuits. Since the blue rectangle in Fig. 132 has area 1, the impulse of f_k in (1) is

$$I_k = \int_0^\infty f_k(t - a)\, dt = \int_a^{a+k} \frac{1}{k}\, dt = 1. \qquad (2)$$

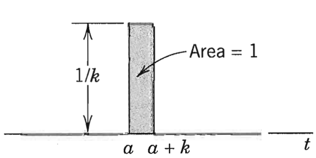

Fig. 132. The function $f_k(t - a)$ in (1)

The sum of this inverse and (7) is the solution of the problem for $0 < t < \pi$, namely (the sines cancel),

$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi. \tag{9}$$

In the second fraction in (6), taken with the minus sign, we have the factor $e^{-\pi s}$, so that from (8) and the second shifting theorem (Sec. 6.3) we get the inverse transform of this fraction for $t > 0$ in the form

$$\begin{aligned} &+2\cos(2t - 2\pi) + \sin(2t - 2\pi) - e^{-(t-\pi)}[2\cos(t-\pi) + 4\sin(t-\pi)] \\ &= 2\cos 2t + \sin 2t + e^{-(t-\pi)}(2\cos t + 4\sin t). \end{aligned}$$

The sum of this and (9) is the solution for $t > \pi$,

$$y(t) = e^{-t}[(3 + 2e^{\pi})\cos t + 4e^{\pi}\sin t] \qquad \text{if } t > \pi. \tag{10}$$

Figure 136 shows (9) (for $0 < t < \pi$) and (10) (for $t > \pi$), a beginning vibration, which goes to zero rapidly because of the damping and the absence of a driving force after $t = \pi$. ■

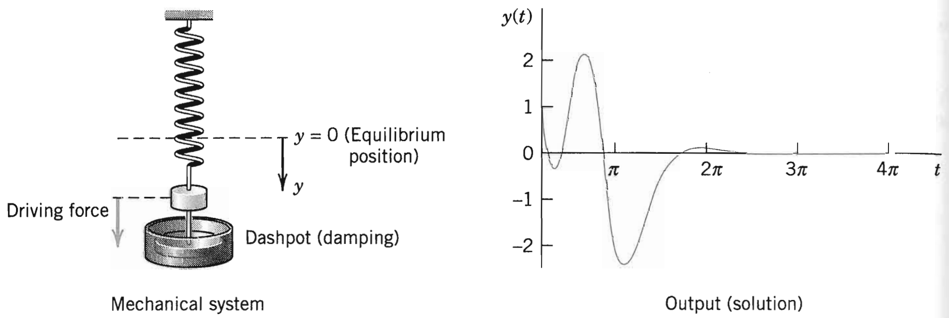

Fig. 136. Example 4

The case of repeated complex factors $[(s - a)(s - \bar{a})]^2$, which is important in connection with resonance, will be handled by "convolution" in the next section.

PROBLEM SET 6.4

1. **CAS PROJECT. Effect of Damping.** Consider a vibrating system of your choice modeled by

$$y'' + cy' + ky = \delta(t).$$

(a) Using graphs of the solution, describe the effect of continuously decreasing the damping to 0, keeping k constant.

(b) What happens if c is kept constant and k is continuously increased, starting from 0?

(c) Extend your results to a system with two δ-functions on the right, acting at different times.

2. **CAS EXPERIMENT. Limit of a Rectangular Wave. Effects of Impulse.**

(a) In Example 1 in the text, take a rectangular wave of area 1 from 1 to $1 + k$. Graph the responses for a sequence of values of k approaching zero, illustrating that for smaller and smaller k those curves approach the curve shown in Fig. 134. *Hint:* If your CAS gives no solution for the differential equation, involving k, take specific k's from the beginning.

(b) Experiment on the response of the ODE in Example 1 (or of another ODE of your choice) to an impulse $\delta(t - a)$ for various systematically chosen a (> 0); choose initial conditions $y(0) \neq 0$, $y'(0) = 0$. Also consider the solution if no impulse is applied. Is there a dependence of the response on a? On b if you choose $b\delta(t - a)$? Would $-\delta(t - \tilde{a})$ with $\tilde{a} > a$ annihilate the effect of $\delta(t - a)$? Can you think of other questions that one could consider experimentally by inspecting graphs?

3–12 EFFECT OF DELTA (IMPULSE) ON VIBRATING SYSTEMS

Find and graph or sketch the solution of the IVP. Show the details.

3. $y'' + 4y = \delta(t - \pi)$, $y(0) = 8, y'(0) = 0$

4. $y'' + 16y = 4\delta(t - 3\pi), \quad y(0) = 2, y'(0) = 0$

5. $y'' + y = \delta(t - \pi) - \delta(t - 2\pi),$
$y(0) = 0, y'(0) = 1$

6. $y'' + 4y' + 5y = \delta(t - 1), \quad y(0) = 0, y'(0) = 3$

7. $4y'' + 24y' + 37y = 17e^{-t} + \delta(t - \frac{1}{2}),$
$y(0) = 1, y'(0) = 1$

8. $y'' + 3y' + 2y = 10(\sin t + \delta(t - 1)), \quad y(0) = 1,$
$y'(0) = -1$

9. $y'' + 4y' + 5y = [1 - u(t - 10)]e^t - e^{10}\delta(t - 10),$
$y(0) = 0, y'(0) = 1$

10. $y'' + 5y' + 6y = \delta(t - \frac{1}{2}\pi) + u(t - \pi)\cos t,$
$y(0) = 0, y'(0) = 0$

11. $y'' + 5y' + 6y = u(t - 1) + \delta(t - 2),$
$y(0) = 0, y'(0) = 1$

12. $y'' + 2y' + 5y = 25t - 100\delta(t - \pi), \quad y(0) = -2,$
$y'(0) = 5$

13. PROJECT. Heaviside Formulas. (a) Show that for a simple root a and fraction $A/(s - a)$ in $F(s)/G(s)$ we have the *Heaviside formula*

$$A = \lim_{s \to a} \frac{(s - a)F(s)}{G(s)}.$$

(b) Similarly, show that for a root a of order m and fractions in

$$\frac{F(s)}{G(s)} = \frac{A_m}{(s - a)^m} + \frac{A_{m-1}}{(s - a)^{m-1}} + \cdots + \frac{A_1}{s - a} + \text{further fractions}$$

we have the *Heaviside formulas* for the first coefficient

$$A_m = \lim_{s \to a} \frac{(s - a)^m F(s)}{G(s)}$$

and for the other coefficients

$$A_k = \frac{1}{(m - k)!} \lim_{s \to a} \frac{d^{m-k}}{ds^{m-k}} \left[\frac{(s - a)^m F(s)}{G(s)} \right], \quad k = 1, \cdots, m - 1.$$

14. TEAM PROJECT. Laplace Transform of Periodic Functions

(a) Theorem. *The Laplace transform of a piecewise continuous function $f(t)$ with period p is*

$$\text{(11)} \qquad \mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt \qquad (s > 0).$$

Prove this theorem. *Hint:* Write $\int_0^\infty = \int_0^p + \int_p^{2p} + \cdots$. Set $t = (n - 1)p$ in the nth integral. Take out $e^{-(n-1)p}$ from under the integral sign. Use the sum formula for the geometric series.

(b) Half-wave rectifier. Using (11), show that the half-wave rectification of $\sin \omega t$ in Fig. 137 has the Laplace transform

$$\mathcal{L}(f) = \frac{\omega(1 + e^{-\pi s/\omega})}{(s^2 + \omega^2)(1 - e^{-2\pi s/\omega})} = \frac{\omega}{(s^2 + \omega^2)(1 - e^{-\pi s/\omega})}.$$

(A *half-wave rectifier* clips the negative portions of the curve. A *full-wave rectifier* converts them to positive; see Fig. 138.)

(c) Full-wave rectifier. Show that the Laplace transform of the full-wave rectification of $\sin \omega t$ is

$$\frac{\omega}{s^2 + \omega^2} \coth \frac{\pi s}{2\omega}.$$

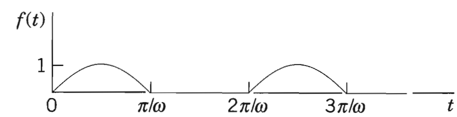

Fig. 137. Half-wave rectification

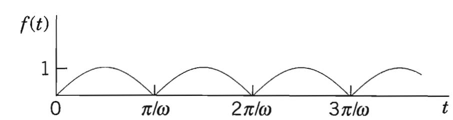

Fig. 138. Full-wave rectification

(d) Saw-tooth wave. Find the Laplace transform of the saw-tooth wave in Fig. 139.

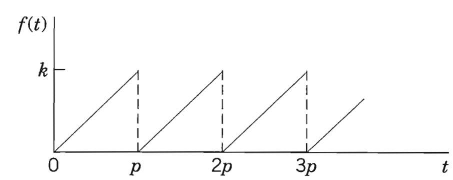

Fig. 139. Saw-tooth wave

15. Staircase function. Find the Laplace transform of the staircase function in Fig. 140 by noting that it is the difference of kt/p and the function in 14(d).

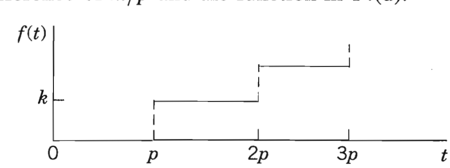

Fig. 140. Staircase function

Solution. By (1) we can write $y - (1 + t) * y = 1 - \sinh t$. Writing $Y = \mathcal{L}(y)$, we obtain by using the convolution theorem and then taking common denominators

$$Y(s)\left[1 - \left(\frac{1}{s} + \frac{1}{s^2}\right)\right] = \frac{1}{s} - \frac{1}{s^2 - 1}, \qquad \text{hence} \qquad Y(s) \cdot \frac{s^2 - s - 1}{s^2} = \frac{s^2 - 1 - s}{s(s^2 - 1)}.$$

$(s^2 - s - 1)/s$ cancels on both sides, so that solving for Y simply gives

$$Y(s) = \frac{s}{s^2 - 1} \qquad \text{and the solution is} \qquad y(t) = \cosh t.$$

PROBLEM SET 6.5

1–7 CONVOLUTIONS BY INTEGRATION

Find:

1. $1 * 1$
2. $1 * \sin \omega t$
3. $e^t * e^{-t}$
4. $(\cos \omega t) * (\cos \omega t)$
5. $(\sin \omega t) * (\cos \omega t)$
6. $e^{at} * e^{bt} \ (a \neq b)$
7. $t * e^t$

8–14 INTEGRAL EQUATIONS

Solve by the Laplace transform, showing the details:

8. $y(t) + 4\int_0^t y(\tau)(t - \tau)\, d\tau = 2t$
9. $y(t) - \int_0^t y(\tau)\, d\tau = 1$
10. $y(t) - \int_0^t y(\tau) \sin 2(t - \tau)\, d\tau = \sin 2t$
11. $y(t) + \int_0^t (t - \tau) y(\tau)\, d\tau = 1$
12. $y(t) + \int_0^t y(\tau) \cosh(t - \tau)\, d\tau = t + e^t$
13. $y(t) + 2e^t \int_0^t y(\tau) e^{-\tau}\, d\tau = te^t$
14. $y(t) - \int_0^t y(\tau)(t - \tau)\, d\tau = 2 - \frac{1}{2}t^2$

15. **CAS EXPERIMENT. Variation of a Parameter.**
(a) Replace 2 in Prob. 13 by a parameter k and investigate graphically how the solution curve changes if you vary k, in particular near $k = -2$.
(b) Make similar experiments with an integral equation of your choice whose solution is oscillating.

16. **TEAM PROJECT. Properties of Convolution.** Prove:
(a) Commutativity, $f * g = g * f$
(b) Associativity, $(f * g) * v = f * (g * v)$
(c) Distributivity, $f * (g_1 + g_2) = f * g_1 + f * g_2$
(d) **Dirac's delta.** Derive the sifting formula (4) in Sec. 6.4 by using f_k with $a = 0$ [(1), Sec. 6.4] and applying the mean value theorem for integrals.
(e) **Unspecified driving force.** Show that forced vibrations governed by

$$y'' + \omega^2 y = r(t), \quad y(0) = K_1, \quad y'(0) = K_2$$

with $\omega \neq 0$ and an unspecified driving force $r(t)$ can be written in convolution form,

$$y = \frac{1}{\omega} \sin \omega t * r(t) + K_1 \cos \omega t + \frac{K_2}{\omega} \sin \omega t.$$

17–26 INVERSE TRANSFORMS BY CONVOLUTION

Showing details, find $f(t)$ if $\mathcal{L}(f)$ equals:

17. $\dfrac{5.5}{(s + 1.5)(s - 4)}$
18. $\dfrac{1}{(s - a)^2}$
19. $\dfrac{2\pi s}{(s^2 + \pi^2)^2}$
20. $\dfrac{9}{s(s + 3)}$
21. $\dfrac{\omega}{s^2(s^2 + \omega^2)}$
22. $\dfrac{e^{-as}}{s(s - 2)}$
23. $\dfrac{40.5}{s(s^2 - 9)}$
24. $\dfrac{240}{(s^2 + 1)(s^2 + 25)}$
25. $\dfrac{18s}{(s^2 + 36)^2}$

26. **Partial Fractions.** Solve Probs. 17, 21, and 23 by partial fraction reduction.

Simplification gives

$$(s - s^2)\frac{dY}{ds} + (n + 1 - s)Y = 0.$$

Separating variables, using partial fractions, integrating (with the constant of integration taken to be zero), and taking exponentials, we get

$$\text{(10*)} \qquad \frac{dY}{Y} = -\frac{n + 1 - s}{s - s^2}\,ds = \left(\frac{n}{s - 1} - \frac{n + 1}{s}\right)ds \qquad \text{and} \qquad Y = \frac{(s - 1)^n}{s^{n+1}}.$$

We write $l_n = \mathcal{L}^{-1}(Y)$ and prove **Rodrigues's formula**

$$\text{(10)} \qquad l_0 = 1, \qquad l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \qquad n = 1, 2, \cdots.$$

These are polynomials because the exponential terms cancel if we perform the indicated differentiations. They are called **Laguerre polynomials** and are usually denoted by L_n (see Problem Set 5.7, but we continue to reserve capital letters for transforms). We prove (10). By Table 6.1 and the first shifting theorem (s-shifting),

$$\mathcal{L}(t^n e^{-t}) = \frac{n!}{(s + 1)^{n+1}}, \qquad \text{hence by (3) in Sec. 6.2} \qquad \mathcal{L}\left\{\frac{d^n}{dt^n}(t^n e^{-t})\right\} = \frac{n!s^n}{(s + 1)^{n+1}}$$

because the derivatives up to the order $n - 1$ are zero at 0. Now make another shift and divide by $n!$ to get [see (10) and then (10*)]

$$\mathcal{L}(l_n) = \frac{(s - 1)^n}{s^{n+1}} = Y. \qquad \blacksquare$$

PROBLEM SET 6.6

1. **REVIEW REPORT. Differentiation and Integration of Functions and Transforms.** Make a draft of these four operations from memory. Then compare your draft with the text and write a 2- to 3-page report on these operations and their significance in applications.

2–11 TRANSFORMS BY DIFFERENTIATION

Showing the details of your work, find $\mathcal{L}(f)$ if $f(t)$ equals:

2. $3t \sinh 4t$
3. $\frac{1}{2}te^{-3t}$
4. $te^{-t}\cos t$
5. $t\cos \omega t$
6. $t^2 \sin 3t$
7. $t^2 \cosh 2t$
8. $te^{-kt}\sin t$
9. $\frac{1}{2}t^2 \sin \pi t$
10. $t^n e^{kt}$
11. $4t\cos\frac{1}{2}\pi t$
12. **CAS PROJECT. Laguerre Polynomials. (a)** Write a CAS program for finding $l_n(t)$ in explicit form from (10). Apply it to calculate $l_0, \cdots, l_{10}$. Verify that $l_0, \cdots, l_{10}$ satisfy Laguerre's differential equation (9).

    (b) Show that

    $$l_n(t) = \sum_{m=0}^{n} \frac{(-1)^m}{m!}\binom{n}{m}t^m$$

    and calculate $l_0, \cdots, l_{10}$ from this formula.

    (c) Calculate $l_0, \cdots, l_{10}$ recursively from $l_0 = 1$, $l_1 = 1 - t$ by

    $$(n + 1)l_{n+1} = (2n + 1 - t)l_n - nl_{n-1}.$$

    (d) A **generating function** (definition in Problem Set 5.2) for the Laguerre polynomials is

    $$\sum_{n=0}^{\infty} l_n(t)x^n = (1 - x)^{-1}e^{tx/(x-1)}.$$

    Obtain $l_0, \cdots, l_{10}$ from the corresponding partial sum of this power series in x and compare the l_n with those in (a), (b), or (c).

13. **CAS EXPERIMENT. Laguerre Polynomials.** Experiment with the graphs of $l_0, \cdots, l_{10}$, finding out empirically how the first maximum, first minimum, $\cdots$ is moving with respect to its location as a function of n. Write a short report on this.

14–20 INVERSE TRANSFORMS

Using differentiation, integration, s-shifting, or convolution, and showing the details, find $f(t)$ if $\mathcal{L}(f)$ equals:

14. $\dfrac{s}{(s^2+16)^2}$

15. $\dfrac{s}{(s^2-9)^2}$

16. $\dfrac{2s+6}{(s^2+6s+10)^2}$

17. $\ln \dfrac{s}{s-1}$

18. $\operatorname{arccot} \dfrac{s}{\pi}$

19. $\ln \dfrac{s^2+1}{(s-1)^2}$

20. $\ln \dfrac{s+a}{s+b}$

6.7 Systems of ODEs

The Laplace transform method may also be used for solving systems of ODEs, as we shall explain in terms of typical applications. We consider a first-order linear system with constant coefficients (as discussed in Sec. 4.1)

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + g_1(t) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + g_2(t). \end{aligned} \tag{1}$$

Writing $Y_1 = \mathcal{L}(y_1)$, $Y_2 = \mathcal{L}(y_2)$, $G_1 = \mathcal{L}(g_1)$, $G_2 = \mathcal{L}(g_2)$, we obtain from (1) in Sec. 6.2 the subsidiary system

$$\begin{aligned} sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + G_1(s) \\ sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + G_2(s). \end{aligned}$$

By collecting the Y_1- and Y_2-terms we have

$$\begin{aligned} (a_{11}-s)Y_1 + \quad a_{12}Y_2 \qquad &= -y_1(0) - G_1(s) \\ a_{21}Y_1 \qquad + (a_{22}-s)Y_2 &= -y_2(0) - G_2(s). \end{aligned} \tag{2}$$

By solving this system algebraically for $Y_1(s), Y_2(s)$ and taking the inverse transform we obtain the solution $y_1 = \mathcal{L}^{-1}(Y_1)$, $y_2 = \mathcal{L}^{-1}(Y_2)$ of the given system (1).

Note that (1) and (2) may be written in vector form (and similarly for the systems in the examples); thus, setting $\mathbf{y} = [y_1 \quad y_2]^\mathsf{T}$, $\mathbf{A} = [a_{jk}]$, $\mathbf{g} = [g_1 \quad g_2]^\mathsf{T}$, $\mathbf{Y} = [Y_1 \quad Y_2]^\mathsf{T}$, $\mathbf{G} = [G_1 \quad G_2]^\mathsf{T}$ we have

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} \qquad \text{and} \qquad (\mathbf{A} - s\mathbf{I})\mathbf{Y} = -\mathbf{y}(0) - \mathbf{G}.$$

EXAMPLE 1 Mixing Problem Involving Two Tanks

Tank T_1 in Fig. 144 initially contains 100 gal of pure water. Tank T_2 initially contains 100 gal of water in which 150 lb of salt are dissolved. The inflow into T_1 is 2 gal/min from T_2 and 6 gal/min containing 6 lb of salt from the outside. The inflow into T_2 is 8 gal/min from T_1. The outflow from T_2 is $2 + 6 = 8$ gal/min, as shown in the figure. The mixtures are kept uniform by stirring. Find and plot the salt contents $y_1(t)$ and $y_2(t)$ in T_1 and T_2, respectively.

Elimination (or Cramer's rule in Sec. 7.7) yields the solution, which we can expand in terms of partial fractions,

$$Y_1 = \frac{(s + \sqrt{3k})(s^2 + 2k) + k(s - \sqrt{3k})}{(s^2 + 2k)^2 - k^2} = \frac{s}{s^2 + k} + \frac{\sqrt{3k}}{s^2 + 3k}$$

$$Y_2 = \frac{(s^2 + 2k)(s - \sqrt{3k}) + k(s + \sqrt{3k})}{(s^2 + 2k)^2 - k^2} = \frac{s}{s^2 + k} - \frac{\sqrt{3k}}{s^2 + 3k}.$$

Hence the solution of our initial value problem is (Fig. 147)

$$y_1(t) = \mathcal{L}^{-1}(Y_1) = \cos \sqrt{k}t + \sin \sqrt{3k}t$$

$$y_2(t) = \mathcal{L}^{-1}(Y_2) = \cos \sqrt{k}t - \sin \sqrt{3k}t.$$

We see that the motion of each mass is harmonic (the system is undamped!), being the superposition of a "slow" oscillation and a "rapid" oscillation. ■

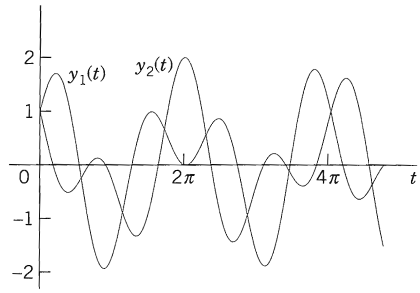

Fig. 147. Solutions in Example 3

PROBLEM SET 6.7

1. TEAM PROJECT. Comparison of Methods for Linear Systems of ODEs

(a) Models. Solve the models in Examples 1 and 2 of Sec. 4.1 by Laplace transforms and compare the amount of work with that in Sec. 4.1. Show the details of your work.

(b) Homogeneous Systems. Solve the systems (8), (11)–(13) in Sec. 4.3 by Laplace transforms. Show the details.

(c) Nonhomogeneous System. Solve the system (3) in Sec. 4.6 by Laplace transforms. Show the details.

2–15 SYSTEMS OF ODES

Using the Laplace transform and showing the details of your work, solve the IVP:

2. $y_1' + y_2 = 0, \quad y_1 + y_2' = 2 \cos t,$
$y_1(0) = 1, \quad y_2(0) = 0$

3. $y_1' = -y_1 + 4y_2, \quad y_2' = 3y_1 - 2y_2,$
$y_1(0) = 3, \quad y_2(0) = 4$

4. $y_1' = 4y_2 - 8 \cos 4t, \quad y_2' = -3y_1 - 9 \sin 4t,$
$y_1(0) = 0, \quad y_2(0) = 3$

5. $y_1' = y_2 + 1 - u(t-1), \quad y_2' = -y_1 + 1 - u(t-1),$
$y_1(0) = 0, \quad y_2(0) = 0$

6. $y_1' = 5y_1 + y_2, \quad y_2' = y_1 + 5y_2,$
$y_1(0) = 1, \quad y_2(0) = -3$

7. $y_1' = 2y_1 - 4y_2 + u(t-1)e^t,$
$y_2' = y_1 - 3y_2 + u(t-1)e^t, \quad y_1(0) = 3, \quad y_2(0) = 0$

8. $y_1' = -2y_1 + 3y_2, \quad y_2' = 4y_1 - y_2,$
$y_1(0) = 4, \quad y_2(0) = 3$

9. $y_1' = 4y_1 + y_2, \quad y_2' = -y_1 + 2y_2, \quad y_1(0) = 3,$
$y_2(0) = 1$

10. $y_1' = -y_2, \quad y_2' = -y_1 + 2[1 - u(t - 2\pi)] \cos t,$
$y_1(0) = 1, \quad y_2(0) = 0$

11. $y_1'' = y_1 + 3y_2, \quad y_2'' = 4y_1 - 4e^t,$
$y_1(0) = 2, \quad y_1'(0) = 3, \quad y_2(0) = 1, \quad y_2'(0) = 2$

12. $y_1'' = -2y_1 + 2y_2, \quad y_2'' = 2y_1 - 5y_2,$
$y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 3, \quad y_2'(0) = 0$

13. $y_1'' + y_2 = -101 \sin 10t, \quad y_2'' + y_1 = 101 \sin 10t,$
$y_1(0) = 0, \quad y_1'(0) = 6, \quad y_2(0) = 8, \quad y_2'(0) = -6$

14. $4y_1' + y_2' - 2y_3' = 0, \quad -2y_1' + y_3' = 1,$
$2y_2' - 4y_3' = -16t$
$y_1(0) = 2, \quad y_2(0) = 0, \quad y_3(0) = 0$

15. $y_1' + y_2' = 2 \sinh t, \quad y_2' + y_3' = e^t,$
$y_3' + y_1' = 2e^t + e^{-t}, \quad y_1(0) = 1, \quad y_2(0) = 1,$
$y_3(0) = 0$

FURTHER APPLICATIONS

16. Forced vibrations of two masses. Solve the model in Example 3 with $k = 4$ and initial conditions $y_1(0) = 1$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2' = -1$ under the assumption that the force $11 \sin t$ is acting on the first body and the force $-11 \sin t$ on the second. Graph the two curves on common axes and explain the motion physically.

17. CAS Experiment. Effect of Initial Conditions. In Prob. 16, vary the initial conditions systematically, describe and explain the graphs physically. The great variety of curves will surprise you. Are they always periodic? Can you find empirical laws for the changes in terms of continuous changes of those conditions?

18. Mixing problem. What will happen in Example 1 if you double all flows (in particular, an increase to 12 gal/min containing 12 lb of salt from the outside), leaving the size of the tanks and the initial conditions as before? First guess, then calculate. Can you relate the new solution to the old one?

19. Electrical network. Using Laplace transforms, find the currents $i_1(t)$ and $i_2(t)$ in Fig. 148, where $v(t) = 390 \cos t$ and $i_1(0) = 0$, $i_2(0) = 0$. How soon will the currents practically reach their steady state?

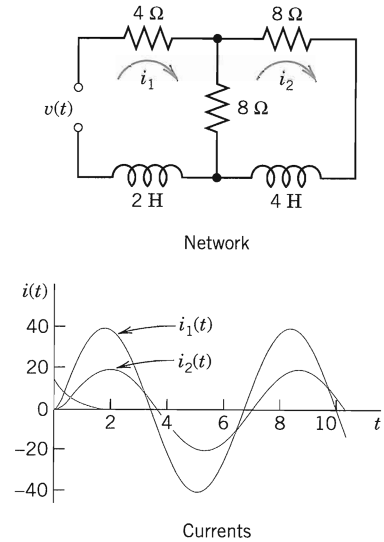

Fig. 148. Electrical network and currents in Problem 19

20. Single cosine wave. Solve Prob. 19 when the EMF (electromotive force) is acting from 0 to 2π only. Can you do this just by looking at Prob. 19, practically without calculation?