Finally, an important application for Cramer's rule dealing with inverse matrices will be given in the next section.

PROBLEM SET 7.7

1–6 GENERAL PROBLEMS

1. **General Properties of Determinants.** Illustrate each statement in Theorems 1 and 2 with an example of your choice.

2. **Second-Order Determinant.** Expand a general second-order determinant in four possible ways and show that the results agree.

3. **Third-Order Determinant.** Do the task indicated in Theorem 2. Also evaluate D by reduction to triangular form.

4. **Expansion Numerically Impractical.** Show that the computation of an nth-order determinant by expansion involves $n!$ multiplications, which if a multiplication takes 10^{-9} sec would take these times:

n	10	15	20	25
Time	0.004 sec	22 min	77 years	$0.5 \cdot 10^9$ years

5. **Multiplication by Scalar.** Show that $\det(k\mathbf{A}) = k^n \det \mathbf{A}$ (not $k \det \mathbf{A}$). Give an example.

6. **Minors, cofactors.** Complete the list in Example 1.

7–15 EVALUATION OF DETERMINANTS

Showing the details, evaluate:

7. $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta \end{vmatrix}$

8. $\begin{vmatrix} 0.4 & 4.9 \\ 1.5 & -1.3 \end{vmatrix}$

9. $\begin{vmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{vmatrix}$

10. $\begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix}$

11. $\begin{vmatrix} 4 & -1 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{vmatrix}$

12. $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

13. $\begin{vmatrix} 0 & 4 & -1 & 5 \\ -4 & 0 & 3 & -2 \\ 1 & -3 & 0 & 1 \\ -5 & 2 & -1 & 0 \end{vmatrix}$

14. $\begin{vmatrix} 4 & 7 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -2 & 2 \end{vmatrix}$

15. $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 2 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 16 \end{vmatrix}$

16. **CAS EXPERIMENT. Determinant of Zeros and Ones.** Find the value of the determinant of the $n \times n$ matrix A_n with main diagonal entries all 0 and all others 1. Try to find a formula for this. Try to prove it by induction. Interpret A_3 and A_4 as *incidence matrices* (as in Problem Set 7.1 but without the minuses) of a triangle and a tetrahedron, respectively; similarly for an *n-simplex*, having n vertices and $n(n-1)/2$ edges (and spanning R^{n-1}, $n = 5, 6, \cdots$).

17–19 RANK BY DETERMINANTS

Find the rank by Theorem 3 (which is not very practical) and check by row reduction. Show details.

17. $\begin{bmatrix} 4 & 9 \\ -8 & -6 \\ 16 & 12 \end{bmatrix}$

18. $\begin{bmatrix} 0 & 4 & -6 \\ 4 & 0 & 10 \\ -6 & 10 & 0 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 5 & 2 & 2 \\ 1 & 3 & 2 & 6 \\ 4 & 0 & 8 & 48 \end{bmatrix}$

20. **TEAM PROJECT. Geometric Applications: Curves and Surfaces Through Given Points.** The idea is to get an equation from the vanishing of the determinant of a homogeneous linear system as the condition for a nontrivial solution in Cramer's theorem. We explain the trick for obtaining such a system for the case of a line L through two given points P_1: (x_1, y_1) and P_2: (x_2, y_2). The unknown line is $ax + by = -c$, say. We write it as $ax + by + c \cdot 1 = 0$. To get a nontrivial solution a, b, c, the determinant of the "coefficients" $x, y, 1$ must be zero. The system is

$$\begin{aligned} &ax + by + c \cdot 1 = 0 \quad (\text{Line } L) \\ (12)\qquad &ax_1 + by_1 + c \cdot 1 = 0 \quad (P_1 \text{ on } L) \\ &ax_2 + by_2 + c \cdot 1 = 0 \quad (P_2 \text{ on } L). \end{aligned}$$

(a) Line through two points. Derive from $D = 0$ in (12) the familiar formula

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}.$$

(b) Plane. Find the analog of (12) for a plane through three given points. Apply it when the points are (1, 1, 1), (3, 2, 6), (5, 0, 5).

(c) Circle. Find a similar formula for a circle in the plane through three given points. Find and sketch the circle through (2, 6), (6, 4), (7, 1).

(d) Sphere. Find the analog of the formula in (c) for a sphere through four given points. Find the sphere through (0, 0, 5), (4, 0, 1), (0, 4, 1), (0, 0, −3) by this formula or by inspection.

(e) General conic section. Find a formula for a general conic section (the vanishing of a determinant of 6th order). Try it out for a quadratic parabola and for a more general conic section of your own choice.

21–25 CRAMER'S RULE

Solve by Cramer's rule. Check by Gauss elimination and back substitution. Show details.

21. $$\begin{aligned} 3x - 5y &= 15.5 \\ 6x + 16y &= 5.0 \end{aligned}$$

22. $$\begin{aligned} 2x - 4y &= -24 \\ 5x + 2y &= 0 \end{aligned}$$

23. $$\begin{aligned} 3y - 4z &= 16 \\ 2x - 5y + 7z &= -27 \\ -x \qquad - 9z &= 9 \end{aligned}$$

24. $$\begin{aligned} 3x - 2y + z &= 13 \\ -2x + y + 4z &= 11 \\ x + 4y - 5z &= -31 \end{aligned}$$

25. $$\begin{aligned} -4w + x + y \qquad &= -10 \\ w - 4x \qquad + z &= 1 \\ w \qquad - 4y + z &= -7 \\ x + y - 4z &= 10 \end{aligned}$$

7.8 Inverse of a Matrix. Gauss–Jordan Elimination

In this section we consider **square** ***matrices exclusively.***

The **inverse** of an $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ is denoted by $\mathbf{A}^{-1}$ and is an $n \times n$ matrix such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{1}$$

where $\mathbf{I}$ is the $n \times n$ unit matrix (see Sec. 7.2).

If $\mathbf{A}$ has an inverse, then $\mathbf{A}$ is called a **nonsingular matrix**. If $\mathbf{A}$ has no inverse, then $\mathbf{A}$ is called a **singular matrix**.

If $\mathbf{A}$ *has an inverse, the inverse is unique.*

Indeed, if both $\mathbf{B}$ and $\mathbf{C}$ are inverses of $\mathbf{A}$, then $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$, so that we obtain the uniqueness from

$$\mathbf{B} = \mathbf{IB} = (\mathbf{CA})\mathbf{B} = \mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C}.$$

We prove next that $\mathbf{A}$ has an inverse (is nonsingular) if and only if it has maximum possible rank n. The proof will also show that $\mathbf{Ax} = \mathbf{b}$ implies $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ provided $\mathbf{A}^{-1}$ exists, and will thus give a motivation for the inverse as well as a relation to linear systems. (But this will ***not*** give a good method of solving $\mathbf{Ax} = \mathbf{b}$ ***numerically*** because the Gauss elimination in Sec. 7.3 requires fewer computations.)

THEOREM 1

Existence of the Inverse

The inverse $\mathbf{A}^{-1}$ *of an* $n \times n$ *matrix* $\mathbf{A}$ *exists if and only if* rank $\mathbf{A} = n$, *thus* (by Theorem 3, Sec. 7.7) *if and only if* det $\mathbf{A} \neq 0$. *Hence* $\mathbf{A}$ *is nonsingular if* rank $\mathbf{A} = n$, *and is singular if* rank $\mathbf{A} < n$.

PROOF If $\mathbf{A}$ or $\mathbf{B}$ is singular, so are $\mathbf{AB}$ and $\mathbf{BA}$ by Theorem 3(c), and (10) reduces to $0 = 0$ by Theorem 3 in Sec. 7.7.

Now let $\mathbf{A}$ and $\mathbf{B}$ be nonsingular. Then we can reduce $\mathbf{A}$ to a diagonal matrix $\hat{\mathbf{A}} = [a_{jk}]$ by Gauss–Jordan steps. Under these operations, det $\mathbf{A}$ retains its value, by Theorem 1 in Sec. 7.7, (a) and (b) [not (c)] except perhaps for a sign reversal in row interchanging when pivoting. But the same operations reduce $\mathbf{AB}$ to $\hat{\mathbf{A}}\mathbf{B}$ with the same effect on det ($\mathbf{AB}$). Hence it remains to prove (10) for $\hat{\mathbf{A}}\mathbf{B}$; written out,

$$\hat{\mathbf{A}}\mathbf{B} = \begin{bmatrix} \hat{a}_{11} & 0 & \cdots & 0 \\ 0 & \hat{a}_{22} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \hat{a}_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{a}_{11}b_{11} & \hat{a}_{11}b_{12} & \cdots & \hat{a}_{11}b_{1n} \\ \hat{a}_{22}b_{21} & \hat{a}_{22}b_{22} & \cdots & \hat{a}_{22}b_{2n} \\ & & \vdots & \\ \hat{a}_{nn}b_{n1} & \hat{a}_{nn}b_{n2} & \cdots & \hat{a}_{nn}b_{nn} \end{bmatrix}.$$

We now take the determinant det ($\hat{\mathbf{A}}\mathbf{B}$). On the right we can take out a factor $\hat{a}_{11}$ from the first row, $\hat{a}_{22}$ from the second, $\cdots$, $\hat{a}_{nn}$ from the nth. But this product $\hat{a}_{11}\hat{a}_{22}\cdots\hat{a}_{nn}$ equals det $\hat{\mathbf{A}}$ because $\hat{\mathbf{A}}$ is diagonal. The remaining determinant is det $\mathbf{B}$. This proves (10) for det ($\mathbf{AB}$), and the proof for det ($\mathbf{BA}$) follows by the same idea. ■

This completes our discussion of linear systems (Secs. 7.3–7.8). Section 7.9 on vector spaces and linear transformations is optional. ***Numeric methods*** are discussed in Secs. 20.1–20.4, which are independent of other sections on numerics.

PROBLEM SET 7.8

1–10 INVERSE

Find the inverse by Gauss–Jordan (or by (4*) if $n = 2$). Check by using (1).

1. $\begin{bmatrix} 1.80 & -2.32 \\ -0.25 & 0.60 \end{bmatrix}$

2. $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

3. $\begin{bmatrix} 0.3 & -0.1 & 0.5 \\ 2 & 6 & 4 \\ 5 & 0 & 9 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 & 0.1 \\ 0 & -0.4 & 0 \\ 2.5 & 0 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 1 \end{bmatrix}$

6. $\begin{bmatrix} -4 & 0 & 0 \\ 0 & 8 & 13 \\ 0 & 3 & 5 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 8 & 0 \\ 0 & 0 & 4 \\ 2 & 0 & 0 \end{bmatrix}$

10. $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$

11–18 SOME GENERAL FORMULAS

11. **Inverse of the square.** Verify $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$ for $\mathbf{A}$ in Prob. 1.

12. Prove the formula in Prob. 11.

13. **Inverse of the transpose.** Verify $(\mathbf{A}^{\mathsf{T}})^{-1} = (\mathbf{A}^{-1})^{\mathsf{T}}$ for $\mathbf{A}$ in Prob. 1.
14. Prove the formula in Prob. 13.
15. **Inverse of the inverse.** Prove that $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
16. **Rotation.** Give an application of the matrix in Prob. 2 that makes the form of the inverse obvious.
17. **Triangular matrix.** Is the inverse of a triangular matrix always triangular (as in Prob. 5)? Give reason.
18. **Row interchange.** Same task as in Prob. 16 for the matrix in Prob. 7.

19–20 FORMULA (4)

Formula (4) is occasionally needed in theory. To understand it, apply it and check the result by Gauss–Jordan:

19. In Prob. 3
20. In Prob. 6

7.9 Vector Spaces, Inner Product Spaces, Linear Transformations *Optional*

We have captured the essence of vector spaces in Sec. 7.4. There we dealt with ***special vector spaces*** that arose quite naturally in the context of matrices and linear systems. The elements of these vector spaces, called *vectors*, satisfied rules (3) and (4) of Sec. 7.1 (which were similar to those for numbers). These special vector spaces were generated by *spans,* that is, linear combination of finitely many vectors. Furthermore, each such vector had n real numbers as *components*. Review this material before going on.

We can generalize this idea by taking *all* vectors with n real numbers as components and obtain the very important ***real n-dimensional vector space*** R^n. The vectors are known as "real vectors." Thus, each vector in R^n is an ordered n-tuple of real numbers.

Now we can consider special values for n. For $n = 2$, we obtain R^2, the vector space of all ordered pairs, which correspond to the **vectors in the plane**. For $n = 3$, we obtain R^3, the vector space of all ordered triples, which are the **vectors in 3-space**. These vectors have wide applications in mechanics, geometry, and calculus and are basic to the engineer and physicist.

Similarly, if we take all ordered n-tuples of *complex numbers* as vectors and complex numbers as scalars, we obtain the **complex vector space** C^n, which we shall consider in Sec. 8.5.

Furthermore, there are other sets of practical interest consisting of matrices, functions, transformations, or others for which addition and scalar multiplication can be defined in an almost natural way so that they too form vector spaces.

It is perhaps not too great an intellectual jump to create, from the ***concrete model*** R^n, the ***abstract concept*** of a ***real vector space*** V by taking the basic properties (3) and (4) in Sec. 7.1 as axioms. In this way, the definition of a real vector space arises.

DEFINITION

Real Vector Space

A nonempty set V of elements $\mathbf{a}, \mathbf{b}, \cdots$ is called a **real vector space** (or *real linear space*), and these elements are called **vectors** (regardless of their nature, which will come out from the context or will be left arbitrary) if, in V, there are defined two algebraic operations (called *vector addition* and *scalar multiplication*) as follows.

I. Vector addition associates with every pair of vectors $\mathbf{a}$ and $\mathbf{b}$ of V a unique vector of V, called the *sum* of $\mathbf{a}$ and $\mathbf{b}$ and denoted by $\mathbf{a} + \mathbf{b}$, such that the following axioms are satisfied.

PROBLEM SET 7.9

1. **Basis.** Find three bases of R^2.
2. **Uniqueness.** Show that the representation $\mathbf{v} = c_1\mathbf{a}_{(1)} + \cdots + c_n\mathbf{a}_{(n)}$ of any given vector in an n-dimensional vector space V in terms of a given basis $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(n)}$ for V is unique. ***Hint.*** Take two representations and consider the difference.

3–10 VECTOR SPACE

(More problems in Problem Set 9.4.) Is the given set, taken with the usual addition and scalar multiplication, a vector space? Give reason. If your answer is yes, find the dimension and a basis.

3. All vectors in R^3 satisfying $-v_1 + 2v_2 + 3v_3 = 0$, $-4v_1 + v_2 + v_3 = 0$.
4. All skew-symmetric 3×3 matrices.
5. All polynomials in x of degree 4 or less with nonnegative coefficients.
6. All functions $y(x) = a \cos 2x + b \sin 2x$ with arbitrary constants a and b.
7. All functions $y(x) = (ax + b)e^{-x}$ with any constant a and b.
8. All $n \times n$ matrices $\mathbf{A}$ with fixed n and $\det \mathbf{A} = 0$.
9. All 2×2 matrices $[a_{jk}]$ with $a_{11} + a_{22} = 0$.
10. All 3×2 matrices $[a_{jk}]$ with first column any multiple of $[3 \quad 0 \quad -5]^\mathsf{T}$.

11–14 LINEAR TRANSFORMATIONS

Find the inverse transformation. Show the details.

11. $y_1 = 0.5x_1 - 0.5x_2$
 $y_2 = 1.5x_1 - 2.5x_2$
12. $y_1 = 3x_1 + 2x_2$
 $y_2 = 4x_1 + x_2$
13. $y_1 = 5x_1 + 3x_2 - 3x_3$
 $y_2 = 3x_1 + 2x_2 - 2x_3$
 $y_3 = 2x_1 - x_2 + 2x_3$
14. $y_1 = 0.2x_1 - 0.1x_2$
 $y_2 = -0.2x_2 + 0.1x_3$
 $y_3 = 0.1x_1 + 0.1x_3$

15–20 EUCLIDEAN NORM

Find the Euclidean norm of the vectors:

15. $[3 \quad 1 \quad -4]^\mathsf{T}$
16. $[\frac{1}{2} \quad \frac{1}{3} \quad -\frac{1}{2} \quad -\frac{1}{3}]^\mathsf{T}$
17. $[1 \quad 0 \quad 0 \quad 1 \quad -1 \quad 0 \quad -1 \quad 1]^\mathsf{T}$
18. $[-4 \quad 8 \quad -1]^\mathsf{T}$
19. $[\frac{2}{3} \quad \frac{2}{3} \quad \frac{1}{3} \quad 0]^\mathsf{T}$
20. $[\frac{1}{2} \quad -\frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2}]^\mathsf{T}$

21–25 INNER PRODUCT. ORTHOGONALITY

21. **Orthogonality.** For what value(s) of k are the vectors $[2 \quad \frac{1}{2} \quad -4 \quad 0]^\mathsf{T}$ and $[5 \quad k \quad 0 \quad \frac{1}{4}]^\mathsf{T}$ orthogonal?
22. **Orthogonality.** Find all vectors in R^3 orthogonal to $[2 \quad 0 \quad 1]$. Do they form a vector space?
23. **Triangle inequality.** Verify (4) for the vectors in Probs. 15 and 18.
24. **Cauchy–Schwarz inequality.** Verify (3) for the vectors in Probs. 16 and 19.
25. **Parallelogram equality.** Verify (5) for the first two column vectors of the coefficient matrix in Prob. 13.

CHAPTER 7 REVIEW QUESTIONS AND PROBLEMS

1. What properties of matrix multiplication differ from those of the multiplication of numbers?
2. Let $\mathbf{A}$ be a 100×100 matrix and $\mathbf{B}$ a 100×50 matrix. Are the following expressions defined or not? $\mathbf{A} + \mathbf{B}$, $\mathbf{A}^2$, $\mathbf{B}^2$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{AA}^\mathsf{T}$, $\mathbf{B}^\mathsf{T}\mathbf{A}$, $\mathbf{B}^\mathsf{T}\mathbf{B}$, $\mathbf{BB}^\mathsf{T}$, $\mathbf{B}^\mathsf{T}\mathbf{AB}$. Give reasons.
3. Are there any linear systems without solutions? With one solution? With more than one solution? Give simple examples.
4. Let $\mathbf{C}$ be 10×10 matrix and $\mathbf{a}$ a column vector with 10 components. Are the following expressions defined or not? $\mathbf{Ca}$, $\mathbf{C}^\mathsf{T}\mathbf{a}$, $\mathbf{Ca}^\mathsf{T}$, $\mathbf{aC}$, $\mathbf{a}^\mathsf{T}\mathbf{C}$, $(\mathbf{Ca}^\mathsf{T})^\mathsf{T}$.
5. Motivate the definition of matrix multiplication.
6. Explain the use of matrices in linear transformations.
7. How can you give the rank of a matrix in terms of row vectors? Of column vectors? Of determinants?
8. What is the role of rank in connection with solving linear systems?
9. What is the idea of Gauss elimination and back substitution?
10. What is the inverse of a matrix? When does it exist? How would you determine it?