PROBLEM SET 8.1

1–16 EIGENVALUES, EIGENVECTORS

Find the eigenvalues. Find the corresponding eigenvectors. Use the given λ or factor in Probs. 11 and 15.

1. $\begin{bmatrix} 3.0 & 0 \\ 0 & -0.6 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

9. $\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$

10. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

11. $\begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}, \quad \lambda = 3$

12. $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$

14. $\begin{bmatrix} 2 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 \end{bmatrix}$

15. $\begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & -1 & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}, \quad (\lambda + 1)^2$

16. $\begin{bmatrix} -3 & 0 & 4 & 2 \\ 0 & 1 & -2 & 4 \\ 2 & 4 & -1 & -2 \\ 0 & 2 & -2 & 3 \end{bmatrix}$

17–20 LINEAR TRANSFORMATIONS AND EIGENVALUES

Find the matrix **A** in the linear transformation $\mathbf{y} = \mathbf{Ax}$, where $\mathbf{x} = [x_1 \ \ x_2]^{\mathsf{T}}$ ($\mathbf{x} = [x_1 \ \ x_2 \ \ x_3]^{\mathsf{T}}$) are Cartesian coordinates. Find the eigenvalues and eigenvectors and explain their geometric meaning.

17. Counterclockwise rotation through the angle $\pi/2$ about the origin in R^2.

18. Reflection about the x_1-axis in R^2.

19. Orthogonal projection (perpendicular projection) of R^2 onto the x_2-axis.

20. Orthogonal projection of R^3 onto the plane $x_2 = x_1$.

21–25 GENERAL PROBLEMS

21. Nonzero defect. Find further 2×2 and 3×3 matrices with positive defect. See Example 3.

22. Multiple eigenvalues. Find further 2×2 and 3×3 matrices with multiple eigenvalues. See Example 2.

23. Complex eigenvalues. Show that the eigenvalues of a real matrix are real or complex conjugate in pairs.

24. Inverse matrix. Show that $\mathbf{A}^{-1}$ exists if and only if the eigenvalues $\lambda_1, \cdots, \lambda_n$ are all nonzero, and then $\mathbf{A}^{-1}$ has the eigenvalues $1/\lambda_1, \cdots, 1/\lambda_n$.

25. Transpose. Illustrate Theorem 3 with examples of your own.

8.2 Some Applications of Eigenvalue Problems

We have selected some typical examples from the wide range of applications of matrix eigenvalue problems. The last example, that is, Example 4, shows an application involving vibrating springs and ODEs. It falls into the domain of Chapter 4, which covers matrix eigenvalue problems related to ODE's modeling mechanical systems and electrical

We try a vector solution of the form

$$\mathbf{y} = \mathbf{x}e^{\omega t}. \tag{8}$$

This is suggested by a mechanical system of a single mass on a spring (Sec. 2.4), whose motion is given by exponential functions (and sines and cosines). Substitution into (7) gives

$$\omega^2 \mathbf{x} e^{\omega t} = \mathbf{A}\mathbf{x}e^{\omega t}.$$

Dividing by $e^{\omega t}$ and writing $\omega^2 = \lambda$, we see that our mechanical system leads to the eigenvalue problem

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x} \qquad \text{where } \lambda = \omega^2. \tag{9}$$

From Example 1 in Sec. 8.1 we see that $\mathbf{A}$ has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -6$. Consequently, $\omega = \pm\sqrt{-1} = \pm i$ and $\sqrt{-6} = \pm i\sqrt{6}$, respectively. Corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \tag{10}$$

From (8) we thus obtain the four complex solutions [see (10), Sec. 2.2]

$$\mathbf{x}_1 e^{\pm it} = \mathbf{x}_1(\cos t \pm i \sin t),$$
$$\mathbf{x}_2 e^{\pm i\sqrt{6}t} = \mathbf{x}_2(\cos \sqrt{6}\, t \pm i \sin \sqrt{6}\, t).$$

By addition and subtraction (see Sec. 2.2) we get the four real solutions

$$\mathbf{x}_1 \cos t, \qquad \mathbf{x}_1 \sin t, \qquad \mathbf{x}_2 \cos \sqrt{6}\, t, \qquad \mathbf{x}_2 \sin \sqrt{6}\, t.$$

A general solution is obtained by taking a linear combination of these,

$$\mathbf{y} = \mathbf{x}_1(a_1 \cos t + b_1 \sin t) + \mathbf{x}_2(a_2 \cos \sqrt{6}\, t + b_2 \sin \sqrt{6}\, t)$$

with arbitrary constants a_1, b_1, a_2, b_2 (to which values can be assigned by prescribing initial displacement and initial velocity of each of the two masses). By (10), the components of $\mathbf{y}$ are

$$y_1 = a_1 \cos t + b_1 \sin t + 2a_2 \cos \sqrt{6}\, t + 2b_2 \sin \sqrt{6}\, t$$
$$y_2 = 2a_1 \cos t + 2b_1 \sin t - a_2 \cos \sqrt{6}\, t - b_2 \sin \sqrt{6}\, t.$$

These functions describe harmonic oscillations of the two masses. Physically, this had to be expected because we have neglected damping. ■

PROBLEM SET 8.2

1–6 ELASTIC DEFORMATIONS

Given $\mathbf{A}$ in a deformation $\mathbf{y} = \mathbf{A}\mathbf{x}$, find the principal directions and corresponding factors of extension or contraction. Show the details.

1. $\begin{bmatrix} 3.0 & 1.5 \\ 1.5 & 3.0 \end{bmatrix}$

2. $\begin{bmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{bmatrix}$

3. $\begin{bmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$

4. $\begin{bmatrix} 5 & 2 \\ 2 & 13 \end{bmatrix}$

5. $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$

6. $\begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}$

7–9 MARKOV PROCESSES

Find the limit state of the Markov process modeled by the given matrix. Show the details.

7. $\begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$

8. $\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix}$

9. $\begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.4 & 0.1 & 0.4 \\ 0 & 0.8 & 0.4 \end{bmatrix}$

10–12 AGE-SPECIFIC POPULATION

Find the growth rate in the Leslie model (see Example 3) with the matrix as given. Show the details.

10. $\begin{bmatrix} 0 & 9.0 & 5.0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$ **11.** $\begin{bmatrix} 0 & 3.45 & 0.60 \\ 0.90 & 0 & 0 \\ 0 & 0.45 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 3.0 & 2.0 & 2.0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$

13–15 LEONTIEF MODELS[1]

13. Leontief input–output model. Suppose that three industries are interrelated so that their outputs are used as inputs by themselves, according to the 3×3 **consumption matrix**

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} 0.1 & 0.5 & 0 \\ 0.8 & 0 & 0.4 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}$$

where a_{jk} is the fraction of the output of industry k consumed (purchased) by industry j. Let p_j be the price charged by industry j for its total output. A problem is to find prices so that for each industry, total expenditures equal total income. Show that this leads to $\mathbf{Ap} = \mathbf{p}$, where $\mathbf{p} = [p_1 \;\; p_2 \;\; p_3]^\mathsf{T}$, and find a solution $\mathbf{p}$ with nonnegative p_1, p_2, p_3.

14. Show that a consumption matrix as considered in Prob. 13 must have column sums 1 and always has the eigenvalue 1.

15. Open Leontief input–output model. If not the whole output but only a portion of it is consumed by the industries themselves, then instead of $\mathbf{Ax} = \mathbf{x}$ (as in Prob. 13), we have $\mathbf{x} - \mathbf{Ax} = \mathbf{y}$, where $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^\mathsf{T}$ is produced, $\mathbf{Ax}$ is consumed by the industries, and, thus, $\mathbf{y}$ is the net production available for other consumers. Find for what production $\mathbf{x}$ a given **demand vector** $\mathbf{y} = [0.1 \;\; 0.3 \;\; 0.1]^\mathsf{T}$ can be achieved if the consumption matrix is

$$\mathbf{A} = \begin{bmatrix} 0.1 & 0.4 & 0.2 \\ 0.5 & 0 & 0.1 \\ 0.1 & 0.4 & 0.4 \end{bmatrix}.$$

16–20 GENERAL PROPERTIES OF EIGENVALUE PROBLEMS

Let $\mathbf{A} = [a_{jk}]$ be an $n \times n$ matrix with (not necessarily distinct) eigenvalues $\lambda_1, \cdots, \lambda_n$. Show.

16. Trace. The sum of the main diagonal entries, called the *trace* of $\mathbf{A}$, equals the sum of the eigenvalues of $\mathbf{A}$.

17. "Spectral shift." $\mathbf{A} - k\mathbf{I}$ has the eigenvalues $\lambda_1 - k, \cdots, \lambda_n - k$ and the same eigenvectors as $\mathbf{A}$.

18. Scalar multiples, powers. $k\mathbf{A}$ has the eigenvalues $k\lambda_1, \cdots, k\lambda_n$. $\mathbf{A}^m (m = 1, 2, \cdots)$ has the eigenvalues $\lambda_1^m, \cdots, \lambda_n^m$. The eigenvectors are those of $\mathbf{A}$.

19. Spectral mapping theorem. The **"polynomial matrix"**

$$p(\mathbf{A}) = k_m\mathbf{A}^m + k_{m-1}\mathbf{A}^{m-1} + \cdots + k_1\mathbf{A} + k_0\mathbf{I}$$

has the eigenvalues

$$p(\lambda_j) = k_m\lambda_j^m + k_{m-1}\lambda_j^{m-1} + \cdots + k_1\lambda_j + k_0$$

where $j = 1, \cdots, n$, and the same eigenvectors as $\mathbf{A}$.

20. Perron's theorem. A Leslie matrix $\mathbf{L}$ with positive $l_{12}, l_{13}, l_{21}, l_{32}$ has a positive eigenvalue. (This is a special case of the Perron–Frobenius theorem in Sec. 20.7, which is difficult to prove in its general form.)

8.3 Symmetric, Skew-Symmetric, and Orthogonal Matrices

We consider three classes of real square matrices that, because of their remarkable properties, occur quite frequently in applications. The first two matrices have already been mentioned in Sec. 7.2. The goal of Sec. 8.3 is to show their remarkable properties.

[1]WASSILY LEONTIEF (1906–1999). American economist at New York University. For his input–output analysis he was awarded the Nobel Prize in 1973.

PROOF The first part of the statement holds for any real matrix **A** because its characteristi polynomial has real coefficients, so that its zeros (the eigenvalues of **A**) must be a indicated. The claim that $|\lambda| = 1$ will be proved in Sec. 8.5. ∎

EXAMPLE 5 **Eigenvalues of an Orthogonal Matrix**

The orthogonal matrix in Example 1 has the characteristic equation

$$-\lambda^3 + \tfrac{2}{3}\lambda^2 + \tfrac{2}{3}\lambda - 1 = 0.$$

Now one of the eigenvalues must be real (why?), hence $+1$ or -1. Trying, we find -1. Division by $\lambda +$ gives $-(\lambda^2 - 5\lambda/3 + 1) = 0$ and the two eigenvalues $(5 + i\sqrt{11})/6$ and $(5 - i\sqrt{11})/6$, which have absolut value 1. Verify all of this. ∎

Looking back at this section, you will find that the numerous basic results it contains hav relatively short, straightforward proofs. This is typical of large portions of matri eigenvalue theory.

PROBLEM SET 8.3

1–10 SPECTRUM

Are the following matrices symmetric, skew-symmetric, or orthogonal? Find the spectrum of each, thereby illustrating Theorems 1 and 5. Show your work in detail.

1. $\begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$ **2.** $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

3. $\begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix}$ **4.** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

5. $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix}$ **6.** $\begin{bmatrix} a & k & k \\ k & a & k \\ k & k & a \end{bmatrix}$

7. $\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$ **8.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

9. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ **10.** $\begin{bmatrix} \frac{4}{9} & \frac{8}{9} & \frac{1}{9} \\ -\frac{7}{9} & \frac{4}{9} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{1}{9} & \frac{8}{9} \end{bmatrix}$

11. WRITING PROJECT. Section Summary. Summarize the main concepts and facts in this section, giving illustrative examples of your own.

12. CAS EXPERIMENT. Orthogonal Matrices.

(a) Products. Inverse. Prove that the product of two orthogonal matrices is orthogonal, and so is the inverse of an orthogonal matrix. What does this mean in terms of rotations?

(b) Rotation. Show that (6) is an orthogonal trans formation. Verify that it satisfies Theorem 3. Find the inverse transformation.

(c) Powers. Write a program for computing power $\mathbf{A}^m$ $(m = 1, 2, \cdots)$ of a 2×2 matrix **A** and thei spectra. Apply it to the matrix in Prob. 1 (call it **A**). To what rotation does **A** correspond? Do the eigenvalue of $\mathbf{A}^m$ have a limit as $m \rightarrow \infty$?

(d) Compute the eigenvalues of $(0.9\mathbf{A})^m$, where **A** i the matrix in Prob. 1. Plot them as points. What is thei limit? Along what kind of curve do these point approach the limit?

(e) Find **A** such that $\mathbf{y} = \mathbf{Ax}$ is a counterclockwis rotation through 30° in the plane.

13–20 GENERAL PROPERTIES

13. Verification. Verify the statements in Example 1.

14. Verify the statements in Examples 3 and 4.

15. Sum. Are the eigenvalues of $\mathbf{A} + \mathbf{B}$ sums of th eigenvalues of **A** and of **B**?

16. Orthogonality. Prove that eigenvectors of a symmetri matrix corresponding to different eigenvalues ar orthogonal. Give examples.

17. Skew-symmetric matrix. Show that the inverse of skew-symmetric matrix is skew-symmetric.

18. Do there exist nonsingular skew-symmetric $n \times n$ matrices with odd n?

19. Orthogonal matrix. Do there exist skew-symmetri orthogonal 3×3 matrices?

20. Symmetric matrix. Do there exist nondiagona symmetric 3×3 matrices that are orthogonal?

PROBLEM SET 8.4

1–5 SIMILAR MATRICES HAVE EQUAL EIGENVALUES

Verify this for $\mathbf{A}$ and $\mathbf{A} = \mathbf{P}^{-1}\mathbf{AP}$. If $\mathbf{y}$ is an eigenvector of $\mathbf{P}$, show that $\mathbf{x} = \mathbf{Py}$ are eigenvectors of $\mathbf{A}$. Show the details of your work.

1. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$

2. $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 7 & -5 \\ 10 & -7 \end{bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0.28 & 0.96 \\ -0.96 & 0.28 \end{bmatrix}$

4. $\mathbf{A} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix},$
$\lambda_1 = 3$

5. $\mathbf{A} = \begin{bmatrix} -5 & 0 & 15 \\ 3 & 4 & -9 \\ -5 & 0 & 15 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. **PROJECT. Similarity of Matrices.** Similarity is basic, for instance, in designing numeric methods.

   (a) Trace. By definition, the **trace** of an $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the sum of the diagonal entries,

   $$\text{trace } \mathbf{A} = a_{11} + a_{22} + \cdots + a_{nn}.$$

   Show that the trace equals the sum of the eigenvalues, each counted as often as its algebraic multiplicity indicates. Illustrate this with the matrices $\mathbf{A}$ in Probs. 1, 3, and 5.

   (b) Trace of product. Let $\mathbf{B} = [b_{jk}]$ be $n \times n$. Show that similar matrices have equal traces, by first proving

   $$\text{trace } \mathbf{AB} = \sum_{i=1}^{n} \sum_{l=1}^{n} a_{il}b_{li} = \text{trace } \mathbf{BA}.$$

   (c) Find a relationship between $\hat{\mathbf{A}}$ in (4) and $\hat{\mathbf{A}} = \mathbf{PAP}^{-1}$.

   (d) Diagonalization. What can you do in (5) if you want to change the order of the eigenvalues in $\mathbf{D}$, for instance, interchange $d_{11} = \lambda_1$ and $d_{22} = \lambda_2$?

7. **No basis.** Find further 2×2 and 3×3 matrices without eigenbasis.

8. **Orthonormal basis.** Illustrate Theorem 2 with further examples.

9–16 DIAGONALIZATION OF MATRICES

Find an eigenbasis (a basis of eigenvectors) and diagonalize. Show the details.

9. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

11. $\begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

12. $\begin{bmatrix} -4.3 & 7.7 \\ 1.3 & 9.3 \end{bmatrix}$

13. $\begin{bmatrix} 4 & 0 & 0 \\ 12 & -2 & 0 \\ 21 & -6 & 1 \end{bmatrix}$

14. $\begin{bmatrix} -5 & -6 & 6 \\ -9 & -8 & 12 \\ -12 & -12 & 16 \end{bmatrix}, \quad \lambda_1 = -2$

15. $\begin{bmatrix} 4 & 3 & 3 \\ 3 & 6 & 1 \\ 3 & 1 & 6 \end{bmatrix}, \quad \lambda_1 = 10$

16. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

17–23 PRINCIPAL AXES. CONIC SECTIONS

What kind of conic section (or pair of straight lines) is given by the quadratic form? Transform it to principal axes. Express $\mathbf{x}^\mathsf{T} = [x_1 \;\; x_2]$ in terms of the new coordinate vector $\mathbf{y}^\mathsf{T} = [y_1 \;\; y_2]$, as in Example 6.

17. $7x_1^2 + 6x_1x_2 + 7x_2^2 = 200$

18. $3x_1^2 + 8x_1x_2 - 3x_2^2 = 10$

19. $3x_1^2 + 22x_1x_2 + 3x_2^2 = 0$

20. $9x_1^2 + 6x_1x_2 + x_2^2 = 10$

21. $x_1^2 - 12x_1x_2 + x_2^2 = 70$

22. $4x_1^2 + 12x_1x_2 + 13x_2^2 = 16$

23. $-11x_1^2 + 84x_1x_2 + 24x_2^2 = 156$

24. Definiteness. A quadratic form $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ and its (symmetric!) matrix **A** are called **(a) positive definite** if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, **(b) negative definite** if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$, **(c) indefinite** if $Q(\mathbf{x})$ takes both positive and negative values. (See Fig. 162.) [$Q(\mathbf{x})$ and **A** are called *positive semidefinite* (*negative semidefinite*) if $Q(\mathbf{x}) \geqq 0$ ($Q(\mathbf{x}) \leqq 0$) for all **x**.] Show that a necessary and sufficient condition for (a), (b), and (c) is that the eigenvalues of **A** are (a) all positive, (b) all negative, and (c) both positive and negative.

Hint. Use Theorem 5.

25. Definiteness. A necessary and sufficient condition for positive definiteness of a quadratic form $Q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ with *symmetric* matrix **A** is that all the **principal minors** are positive (see Ref. [B3], vol. 1, p. 306), that is,

$$a_{11} > 0, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0,$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} > 0, \qquad \cdots, \qquad \det \mathbf{A} > 0.$$

Show that the form in Prob. 22 is positive definite, whereas that in Prob. 23 is indefinite.

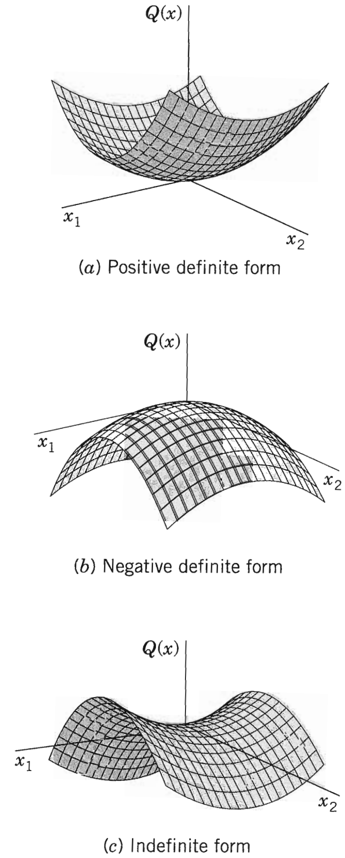

Fig. 162. Quadratic forms in two variables (Problem 24)

8.5 Complex Matrices and Forms. *Optional*

The three classes of matrices in Sec. 8.3 have complex counterparts which are of practical interest in certain applications, for instance, in quantum mechanics. This is mainly because of their spectra as shown in Theorem 1 in this section. The second topic is about extending quadratic forms of Sec. 8.4 to complex numbers. (The reader who wants to brush up on complex numbers may want to consult Sec. 13.1.)

Notations

$\overline{\mathbf{A}} = [\overline{a}_{jk}]$ is obtained from $\mathbf{A} = [a_{jk}]$ by replacing each entry $a_{jk} = \alpha + i\beta$ (α, β real) with its complex conjugate $\overline{a}_{jk} = \alpha - i\beta$. Also, $\overline{\mathbf{A}}^T = [\overline{a}_{kj}]$ is the transpose of $\overline{\mathbf{A}}$, hence the conjugate transpose of **A**.

EXAMPLE 1 Notations

$$\text{If } \mathbf{A} = \begin{bmatrix} 3+4i & 1-i \\ 6 & 2-5i \end{bmatrix}, \quad \text{then } \overline{\mathbf{A}} = \begin{bmatrix} 3-4i & 1+i \\ 6 & 2+5i \end{bmatrix} \quad \text{and } \overline{\mathbf{A}}^T = \begin{bmatrix} 3-4i & 6 \\ 1+i & 2+5i \end{bmatrix}. \quad \blacksquare$$

Hermitian and Skew-Hermitian Forms

The concept of a quadratic form (Sec. 8.4) can be extended to complex. We call the numerator $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\mathbf{x}$ in (1) a **form** in the components $x_1, \cdots, x_n$ of $\mathbf{x}$, which may now be complex. This form is again a sum of n^2 terms

$$
\begin{aligned}
\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\mathbf{x} &= \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\bar{x}_j x_k \\
&= \quad a_{11}\bar{x}_1 x_1 + \cdots + a_{1n}\bar{x}_1 x_n \\
&\quad + a_{21}\bar{x}_2 x_1 + \cdots + a_{2n}\bar{x}_2 x_n \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + a_{n1}\bar{x}_n x_1 + \cdots + a_{nn}\bar{x}_n x_n.
\end{aligned} \tag{7}
$$

$\mathbf{A}$ is called its **coefficient matrix**. The form is called a **Hermitian** or **skew-Hermitian form** if $\mathbf{A}$ is Hermitian or skew-Hermitian, respectively. *The value of a Hermitian form is real, and that of a skew-Hermitian form is pure imaginary or zero.* This can be seen directly from (2) and (3) and accounts for the importance of these forms in physics. Note that (2) and (3) are valid for any vectors because, in the proof of (2) and (3), we did not use that $\mathbf{x}$ is an eigenvector but only that $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{x}$ is real and not 0.

EXAMPLE 6 Hermitian Form

For $\mathbf{A}$ in Example 2 and, say, $\mathbf{x} = [1 + i \quad 5i]^{\mathsf{T}}$ we get

$$\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\mathbf{x} = [1 - i \quad -5i]\begin{bmatrix} 4 & 1 - 3i \\ 1 + 3i & 7 \end{bmatrix}\begin{bmatrix} 1 + i \\ 5i \end{bmatrix} = [1 - i \quad -5i]\begin{bmatrix} 4(1 + i) + (1 - 3i)\cdot 5i \\ (1 + 3i)(1 + i) + 7\cdot 5i \end{bmatrix} = 223.$$

Clearly, if $\mathbf{A}$ and $\mathbf{x}$ in (4) are real, then (7) reduces to a quadratic form, as discussed in the last section.

PROBLEM SET 8.5

1–6 EIGENVALUES AND VECTORS

Is the given matrix Hermitian? Skew-Hermitian? Unitary? Find its eigenvalues and eigenvectors.

1. $\begin{bmatrix} 6 & i \\ -i & 6 \end{bmatrix}$

2. $\begin{bmatrix} i & 1 + i \\ -1 + i & 0 \end{bmatrix}$

3. $\begin{bmatrix} \frac{1}{2} & i\sqrt{\frac{3}{4}} \\ i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}$

4. $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

5. $\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 2 + 2i & 0 \\ 2 - 2i & 0 & 2 + 2i \\ 0 & 2 - 2i & 0 \end{bmatrix}$

7. Pauli spin matrices. Find the eigenvalues and eigenvectors of the so-called *Pauli spin matrices* and show that $\mathbf{S}_x\mathbf{S}_y = i\mathbf{S}_z$, $\mathbf{S}_y\mathbf{S}_x = -i\mathbf{S}_z$, $\mathbf{S}_x^2 = \mathbf{S}_y^2 = \mathbf{S}_z^2 = \mathbf{I}$, where

$$\mathbf{S}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{S}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

$$\mathbf{S}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

8. Eigenvectors. Find eigenvectors of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ in Examples 2 and 3.

9–12 COMPLEX FORMS

Is the matrix **A** Hermitian or skew-Hermitian? Find $\bar{\mathbf{x}}^{\mathsf{T}}\mathbf{A}\mathbf{x}$. Show the details.

9. $\mathbf{A} = \begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -4i \\ 2+2i \end{bmatrix}$

10. $\mathbf{A} = \begin{bmatrix} i & -2+3i \\ 2+3i & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2i \\ 8 \end{bmatrix}$

11. $\mathbf{A} = \begin{bmatrix} i & 1 & 2+i \\ -1 & 0 & 3i \\ -2+i & 3i & i \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix}$

12. $\mathbf{A} = \begin{bmatrix} 1 & i & 4 \\ -i & 3 & 0 \\ 4 & 0 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix}$

13–20 GENERAL PROBLEMS

13. Product. Show that $\overline{(\mathbf{ABC})}^{\mathsf{T}} = -\mathbf{C}^{-1}\mathbf{BA}$ for any $n \times n$ Hermitian **A**, skew-Hermitian **B**, and unitary **C**.

14. Product. Show $\overline{(\mathbf{BA})}^{\mathsf{T}} = -\mathbf{AB}$ for **A** and **B** in Example 2. For any $n \times n$ Hermitian **A** and skew-Hermitian **B**.

15. Decomposition. Show that any square matrix may be written as the sum of a Hermitian and a skew-Hermitian matrix. Give examples.

16. Unitary matrices. Prove that the product of two unitary $n \times n$ matrices and the inverse of a unitary matrix are unitary. Give examples.

17. Powers of unitary matrices in applications may sometimes be very simple. Show that $\mathbf{C}^{12} = \mathbf{I}$ in Example 2. Find further examples.

18. Normal matrix. This important concept denotes a matrix that commutes with its conjugate transpose, $\mathbf{A}\bar{\mathbf{A}}^{\mathsf{T}} = \bar{\mathbf{A}}^{\mathsf{T}}\mathbf{A}$. Prove that Hermitian, skew-Hermitian, and unitary matrices are normal. Give corresponding examples of your own.

19. Normality criterion. Prove that **A** is normal if and only if the Hermitian and skew-Hermitian matrices in Prob. 18 commute.

20. Find a simple matrix that is not normal. Find a normal matrix that is not Hermitian, skew-Hermitian, or unitary.

CHAPTER 8 REVIEW QUESTIONS AND PROBLEMS

1. In solving an eigenvalue problem, what is given and what is sought?
2. Give a few typical applications of eigenvalue problems.
3. Do there exist square matrices without eigenvalues?
4. Can a real matrix have complex eigenvalues? Can a complex matrix have real eigenvalues?
5. Does a 5×5 matrix always have a real eigenvalue?
6. What is algebraic multiplicity of an eigenvalue? Defect?
7. What is an eigenbasis? When does it exist? Why is it important?
8. When can we expect orthogonal eigenvectors?
9. State the definitions and main properties of the three classes of real matrices and of complex matrices that we have discussed.
10. What is diagonalization? Transformation to principal axes?

11–15 SPECTRUM

Find the eigenvalues. Find the eigenvectors.

11. $\begin{bmatrix} 2.5 & 0.5 \\ 0.5 & 2.5 \end{bmatrix}$

12. $\begin{bmatrix} -7 & 4 \\ -12 & 7 \end{bmatrix}$

13. $\begin{bmatrix} 8 & -1 \\ 5 & 2 \end{bmatrix}$

14. $\begin{bmatrix} 7 & 2 & -1 \\ 2 & 7 & 1 \\ -1 & 1 & 8.5 \end{bmatrix}$

15. $\begin{bmatrix} 0 & -3 & -6 \\ 3 & 0 & -6 \\ 6 & 6 & 0 \end{bmatrix}$

16–17 SIMILARITY

Verify that **A** and $\hat{\mathbf{A}} = \mathbf{p}^{-1}\mathbf{AP}$ have the same spectrum.

16. $\mathbf{A} = \begin{bmatrix} 19 & 12 \\ 12 & 1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

17. $\mathbf{A} = \begin{bmatrix} 7 & -4 \\ 12 & -7 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

18. $\mathbf{A} = \begin{bmatrix} -4 & 6 & 6 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$