§11.1 – 11.5 4 prob.

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$f(x)$ on $(-L, L)$

$f(x+2L) = f(x)$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,dx$$

$f(x)$ on $(0, L)$

$$\frac{1}{L}\int_0^L f(x)\,dx \qquad \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\,dx \qquad b_n = 0$$

F-cosine series even extension to $(-L, 0)$

F-sine series odd extension to $(-L, 0)$ $a_0 = 0$, $a_n = 0$

$$f(x) = \begin{cases} 1 & 0 < x < 1/2 \\ 2 & \frac{1}{2} < x < 1 \end{cases} \qquad (0, L), \; L = 1 \qquad a_n = -\frac{2}{n\pi}$$

E-cosine series $\qquad a_0 \overset{?}{=}, \; a_3 = ?$

$$a_0 = \frac{1}{1}\int_0^1 f(x)\,dx = \int_0^{\frac{1}{2}} 1\,dx + \int_{\frac{1}{2}}^1 2\,dx = \frac{1}{2} + 1 = \frac{3}{2}$$

$1, 0, -1, 0$

$2k+1$

$$a_n = \frac{2}{1}\left[\int_0^{\frac{1}{2}} \cos\frac{n\pi x}{1}\,dx + \int_{\frac{1}{2}}^1 2\cos n\pi x\,dx\right]$$

$$= 2\left[\frac{\sin n\pi x}{n\pi}\Big|_0^{\frac{1}{2}} + 2\,\frac{\sin n\pi x}{n\pi}\Big|_{\frac{1}{2}}^1\right]$$

$$= \frac{2}{n\pi}\left[\sin\frac{n\pi}{2} + -2\sin\frac{n\pi}{2}\right] = -2\sin\frac{n\pi}{2}\Big/ n\pi$$

Parseval's Identity

$f(x)$ on $\cancel{(-L, L)}$ $(-\pi, \pi)$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx} = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$f(x)$ on $(0, \pi)$

$(-\pi, 0)$

$$\int_{-\pi}^{\pi} f = 2\int_0^{\pi} f$$

F-cosine

$$\int_{-\pi}^{\pi} f = 2\int_0^{\pi} f$$

F-sine

#11, 12 in §11.4

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = ?$$

$$\boxed{f(x) = \frac{3}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\sin\frac{n\pi}{2}\cdot\frac{1}{n}\cdot\right)\cancel{\sin n\pi x}\ \cos n\pi x}$$

$$a_0 = \frac{3}{2}, \quad a_3 = \frac{2}{3\pi}$$

$$\frac{1}{L}\int_{-L}^{L} f^2 = 2\int_0^1 f^2 = 2\left[\int_0^{\frac{1}{2}} f^2 + \int_{\frac{1}{2}}^{1} f^2\right] = 2\left[\frac{1}{2} + 2\right] = 5$$

$$2\left(\frac{3}{2}\right)^2 + \sum_{n=1}^{\infty}\left(-\frac{2}{\pi}\sin\frac{n\pi}{2}\cdot\frac{1}{n}\right)^2 = \frac{9}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sin\frac{n}{2}\pi\right)^2$$

$$= \frac{9}{2} + \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$$

$$5 = \frac{9}{2} + \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \left(5 - \frac{9}{2}\right) \frac{\pi^2}{4} = \pi^2/8$$

#7, #13 on P_{503} eigenvalues and eigenfunction.

#12 $y'' - 2y' + (\lambda+1)y = 0$, $y(0)=0$, $y(1)=0$

$0 = s^2 - 2s + \lambda + 1 = (s-1)^2 + \lambda \Rightarrow s = 1 \pm \sqrt{-\lambda}$

$\lambda = 0$ $\quad y = c_1 e^t + c_2 t e^t$ $\quad 0 = c_1$

$0 = y(1) = c_2 e \Rightarrow c_2 = 0 \Rightarrow y(t) \equiv 0$

$\lambda = -\nu^2$ $\quad s = 1 \pm \nu$

$y = c_1 e^{(1+\nu)t} + c_2 e^{(1-\nu)t}$ $\quad y(t) \equiv 0$

$\lambda = \nu^2$ $\quad 0 = y(0) = c_1 + c_2$

$0 = y(1) = c_1 e^{1+\nu} + c_2 e^{1-\nu}$ $\quad \det\begin{bmatrix} 1 & 1 \\ e^{1+\nu} & e^{1-\nu} \end{bmatrix} = e^{1-\nu} - e^{1+\nu} \neq 0$

$\lambda = \nu^2$ $\quad s = 1 \pm \sqrt{-\nu^2} = 1 \pm \nu i$ $\quad e^t \cos \nu t$, $e^t \sin \nu t$

$$y = c_1 e^t \cos \nu t + c_2 e^t \sin \nu t$$

$0 = y(0) = c_1$, $\quad 0 = y(1) = c_2 \cancel{e} \sin \nu \implies \boxed{\sin \nu = 0}$

$\nu = n\pi$, $\quad n = 1, 2, 3, \cdots$

$y_n = e^t \sin n\pi t$ — eigenfunctions

$\lambda_n = \nu_n^2 = (n\pi)^2$ — eigenvalues.