1-D Wave Eq (§12.3)

$$u_{tt} = c^2 u_{xx} \quad \text{in } [0,L]\times[0,+\infty)$$

$$c^2 = \frac{T}{\rho}$$

BCs $u(0,t)=0, \quad u(L,t)=0, \quad t\in[0,+\infty)$

ICs $u(x,0)=f(x), \quad u_t(x,0)=g(x), \quad x\in[0,L]$

$$\lambda_n = \frac{cn\pi}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right)\sin\frac{n\pi x}{L}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

D'Alemberts Solution (§12.4)

$$\begin{cases} u_{tt} = c^2 u_{xx} & (-\infty, +\infty) \times [0, +\infty) \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

1-D Heat Eq. (§12.6)

$$c^2 = \frac{K}{\sigma\rho}$$

$$u_t = c^2 u_{xx} \qquad x \in [0, L], \; t \in [0, +\infty)$$

BCs $\quad u(0,t)=0, \;\; u(L,t)=0, \;\; t \in [0,+\infty)$

ICs $\quad u(x,0) = f(x) \qquad x \in [0, L]$

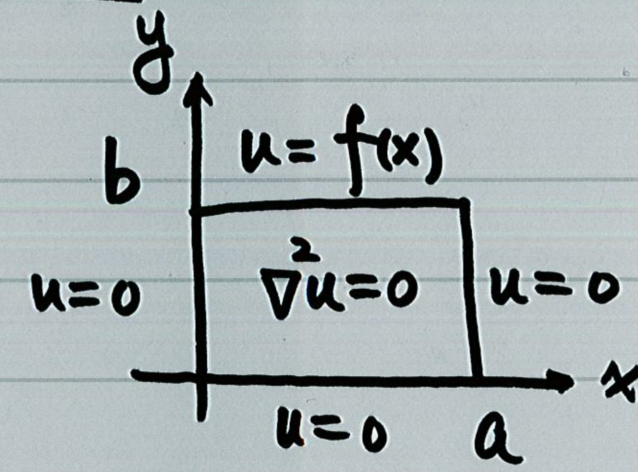
$$\lambda_n = \frac{cn\pi}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-\lambda_n^2 t}$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

2-D Laplace Eq. in a Rectangle $R=[0,a]\times[0,b]$ (§12.6)

$$u_{xx}+u_{yy}=0 \quad \text{in } R$$



BCs $\quad u(0,y)=0, \quad u(a,y)=0$

$\quad u(x,0)=0, \quad u(x,b)=f(x)$

$$\frac{e^{x}-e^{-x}}{2}$$

$$u(x,y)=\sum_{n=1}^{\infty} \underline{A_n^*}\, \sin\frac{n\pi x}{a}\; \underline{\sinh\frac{n\pi y}{b}}$$

$$A_n^*=\frac{2}{a\sinh\frac{n\pi b}{a}}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

2-D Wave Eq. (§12.9)

$$R = [0,a]\times[0,b]$$

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right) \qquad R\times[0,+\infty)$$

BCs $\quad u\big|_{\partial R} = 0$

$$\lambda_{mn}^2 = c^2\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

ICs $\quad u(x,y,0) = f(x,y), \quad u_t(x,y,0) = g(x,y)$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$B_{mn} = \frac{4}{ab}\int_0^b dy\int_0^a dx\; f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \quad B^*_{mn} =$$

2-D Laplace Eq. in Polar Coordinates (§12.10)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{in } \Omega = \left\{(r,\theta) \,\middle|\, 0<r<R,\ -\pi\le\theta\le\pi\right\}$$

BCs $\quad u(R,\theta) = f(\theta)$

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty}\left(a_n\left(\frac{r}{R}\right)^n\cos n\theta + b_n\left(\frac{r}{R}\right)^n\sin n\theta\right)$$

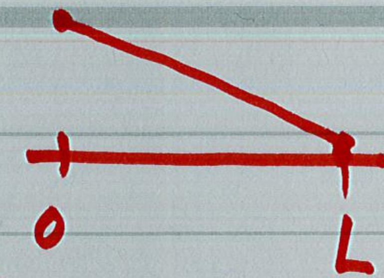
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta$$

$$u_t = c^2 u_{xx} \qquad [0, L]\times[0, +\infty)$$

ICs $u(x,0) = f(x)$

BCs $u(0,t) = \boxed{100}$, $u(L,t) = 0$



$$\boxed{u(0,t)=0, \quad u(L,t)=0}$$

$$u(x,t) = \boxed{\frac{100}{L}(L-x)} + \boxed{W(x,t)}$$

$$W(x,t) = u(x,t) - \frac{100}{L}(L-x)$$

$$W(0,t) = u(0,t) - 100 = 0$$

$$W(L,t) = u(L,t) = 0$$

$$u_t = W_t$$

$$c^2 u_{xx} = c^2 W_{xx}$$

$$W_t = c^2 W_{xx}$$

$$W(0,t) = W(L,t) = 0$$

$$W(x,0) = u(x,0) - \frac{100}{L}(L-x) = \boxed{f(x) - \frac{100}{L}(L-x)}$$

$$w(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

$$B_n = \frac{2}{L}\int_0^L \left[f(x) - \frac{100}{L}(L-x)\right]\sin\frac{n\pi x}{L}\,dx$$

#21 $\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}}\,\frac{1}{1+\omega^2} \Longrightarrow e^{-x} = \mathcal{F}_c^{-1}\left(\underline{\mathcal{F}_c(e^{-x})}\right)$

$$= \underline{\underline{\mathcal{F}_c^{-1}}}\left(\sqrt{\frac{2}{\pi}}\,\frac{1}{1+\omega^2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{1}{1+\omega^2}\cdot\cos\omega x\,d\underline{\underline{\omega}}$$

$$\underline{? = \frac{2}{\pi}\int_0^\infty \frac{\cos 2\omega}{1+\omega^2}\,d\omega}$$

$$= e^{-2}$$

$\mathcal{F}_c(f(x))$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,d\underline{\underline{x}}$$

$$\boxed{e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x}{1+\omega^2}\,d\omega}$$

#22 $\underline{\mathcal{F}\left(e^{-\frac{x^2}{2}}\right) = e^{-\frac{\omega^2}{2}}}$

$$\mathcal{F}\left(x e^{-\frac{x^2}{2}}\right) = -\mathcal{F}\left(\left(e^{-\frac{x^2}{2}}\right)'\right)$$

$$x e^{-\frac{x^2}{2}} = -\left(e^{-\frac{x^2}{2}}\right)'$$

$$= -i\omega\, \mathcal{F}\left(e^{-\frac{x^2}{2}}\right)$$

$$\underline{= -i\omega\, e^{-\frac{\omega^2}{2}}}$$

$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$\mathcal{F}^{-1}(g(\omega)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\, e^{i\omega x}\, d\omega$$

$$\underline{\mathcal{F}(f'(x)) = i\omega\, \mathcal{F}(f(x))}$$

#23 $u_{tt} = u_{xx}$ in $[0,\pi]\times[0,+\infty)$

$u(0,t)=u(\pi,t)=0$

$u(x,0)=f(x)$

$u_t(x,0)=0$

$$f(x)=\underline{\frac{1}{2}\sin 2x} + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{4-(2n+1)^2}\underline{\sin(2n+1)x}$$

$$u(x,t)=\sum_{n=1}^{\infty} B_n \cos nt \sin nx$$

$$u\left(\frac{\pi}{4},\frac{\pi}{2}\right)=$$

$$u(x,0)=\sum_{n=1}^{\infty} B_n \sin nx$$

$$B_{2n+1}=\frac{4}{\pi}\,\frac{(-1)^n}{4-(\ \ )^2}$$

$\|$

$$\sum_{n=1}^{\infty} B_n \boxed{\cos\frac{n\pi}{2}} \sin\frac{n\pi}{4}$$

$B_1=0$, $B_2=\frac{1}{2}$, $\underline{\underline{B_{2n}=0}}$,

$$= B_2\, \cancel{\sin}\, \cos\pi \,\sin\frac{\pi}{2} = -\frac{1}{2}$$