

10 problems (10 pts each) Exams 1 & 2

15 problems (Chapters 11 & 12)
(8 pts each)

11.1-11.9

12.1-12.10 except 12.7

#24

$$u_{tt} = \cancel{4} 4 u_{xx} \quad [0, \pi] \times [0, +\infty)$$

$$c=2$$

$$u(0, t) = u(\pi, t) = 0$$

$$u(x, 0) = \sin x$$

$$u_t(x, 0) = \sin 2x$$

$$u(x, t) = \sum_{n=1}^{\infty} \left(B_n \cos \frac{2n\pi t}{\pi} + B_n^* \sin 2nt \right) \sin nx$$

$$\sin x = u(x, 0) = \sum_{n=1}^{\infty} B_n \sin nx$$

$$u\left(\frac{\pi}{4}, \frac{\pi}{8}\right) = \cos \frac{\pi}{4} \sin \frac{\pi}{4}$$

$$\sin 2x = u_t(x, 0) = \sum_{n=1}^{\infty} 2n B_n^* \cos 2nt \Big|_{t=0} \sin nx$$

$$u(x, t) = \cos 2t \sin x + \frac{1}{4} \sin 4t \sin 2x$$

$$= \sum_{n=1}^{\infty} 2n B_n^* \sin nx$$

$$+ \frac{1}{4} \sin \frac{\pi}{2} \sin \frac{\pi}{2}$$

$$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$\underline{B_1 = 1}, B_n = 0 \text{ for } n \geq 2$$

$$4B_2^* = 1 \Rightarrow B_2^* = \frac{1}{4}$$

$$B_n^* = 0 \text{ for } n \neq 2$$

#25 $u_t = 4 u_{xx} \quad [0, \pi] \times [0, +\infty)$

$$\begin{cases} u(0, t) = u(\pi, t) = 0 \\ u(x, 0) = x(\pi - x) = f(x) \end{cases}$$

$$\underline{x(\pi - x)} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin nx$$

$$u(x, t) = \sum_{n=1}^{\infty} \underline{B_n} \sin \frac{n\pi x}{\pi} e^{-\left(\frac{2n\pi}{\pi}\right)^2 t}$$

$$B_n = \frac{4}{\pi} \cdot \frac{1 - (-1)^n}{n^3}$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin nx e^{-4n^2 t}$$

E.

#26 $u_{tt} = 4 u_{xx} \quad (-\infty, +\infty) \times [0, +\infty) \quad \underline{c=2}$

$$\begin{cases} u(x, 0) = f(x) = \sin x \\ u_t(x, 0) = g(x) = \cos x \end{cases}$$

$$u(x, t) = \frac{1}{2} [f(x+ct) + f(x-ct)]$$

$$+ \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

$$u(0, \frac{\pi}{4}) = \frac{1}{2} \left[\sin \frac{\pi}{2} + \sin \left(-\frac{\pi}{2} \right) \right]$$

$$= \frac{1}{2} [f(x+2t) + f(x-2t)] + \frac{1}{4} \int_{x-2t}^{x+2t} g(s) ds$$

$$x=0$$

$$t = \frac{\pi}{4}$$

$$x+2t = \frac{\pi}{2}$$

$$x-2t = -\frac{\pi}{2}$$

$$+ \frac{1}{4} \sin s \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \neq \frac{1}{4}$$

$$= 0 + \frac{1}{4} \left[\sin \frac{\pi}{2} - \sin \left(-\frac{\pi}{2} \right) \right]$$

$$= \frac{1}{2}$$

#28

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 0 < r < 1, \quad -\pi \leq \theta < \pi$$

$$u(1, \theta) = \cos 2\theta = f(\theta)$$

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\underline{a_n} \left(\frac{r}{R} \right)^n \cos n\theta + \underline{b_n} \left(\frac{r}{R} \right)^n \sin n\theta \right]$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$u(r, \theta) = r^2 \cos 2\theta$$

$$\underline{\cos 2\theta} = u(1, \theta) = \left\{ a_0 + \sum_{n=1}^{\infty} a_n r^n \cos n\theta + b_n r^n \sin n\theta \right\}_{r=1}$$

$$a_2 = 1$$

$$= a_0 + \sum_{n=1}^{\infty} \underline{a_n} \cos n\theta + b_n \sin n\theta$$

$$a_n = 0 \text{ for } n \neq 2, \quad \underline{b_n} = 0$$

#29 $u_t = 4 u_{xx} \quad [0, \pi] \times [0, +\infty)$

$$\left\{ \begin{array}{l} \boxed{u_x(0, t) = 0, u_x(\pi, t) = 0} \end{array} \right.$$

$$u(x, 0) = \sin x$$

$$u(0, t) = u(\pi, t) = 0$$

$$u(0, t) = a, u(\pi, t) = b$$

Given that

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos 2nx$$

$$u(x, t) = F(x) G(t)$$

$$F \dot{G} = 4 F'' G \quad / \quad 4FG$$

$$\frac{F''}{F} = \frac{\dot{G}}{4G} = k$$

$$u_x(x, t) = F'(x) G(t)$$

$$0 = u_x(0, t) = \underline{F'(0)} \underline{G(t)}$$

$$\boxed{F''(x) - k F(x) = 0, F'(0) = 0, F'(\pi) = 0}$$

$$\left[\begin{array}{l} \dot{G}(t) - 4k G(t) = 0 \end{array} \right.$$

$$\begin{cases} F''(x) - k F(x) = 0 \\ F'(0) = 0, F'(\pi) = 0 \end{cases}$$

$$\underline{k = \nu^2 > 0} \quad s^2 - \nu^2 = 0$$

$$\Rightarrow s = \pm \nu$$

$$\Rightarrow F(x) = c_1 e^{\nu x} + c_2 e^{-\nu x}$$

$$F'(x) = c_1 \nu e^{\nu x} - c_2 \nu e^{-\nu x}$$

$$0 = F'(0) = (c_1 - c_2) \nu \Rightarrow \underline{c_1 - c_2 = 0}$$

$$0 = F'(\pi) = \nu [c_1 e^{\nu \pi} - c_2 e^{-\nu \pi}]$$

$$\det A = -e^{-\nu \pi} + e^{\nu \pi} \neq 0 \text{ for } \nu \neq 0$$

$$c_1 = 0, c_2 = 0$$

$$F(x) = 0$$

$$\underline{k=0}$$

$$F = c_1 + c_2 x$$

$$F' = c_2 = 0$$

$$\Rightarrow F(x) = 1$$

$$k = -\nu^2 \quad 0 = s^2 + \nu^2 \Rightarrow s = \pm \nu i$$

$$F(x) = c_1 \cos \nu x + c_2 \sin \nu x$$

$$F'(x) = -c_1 \nu \sin \nu x + c_2 \nu \cos \nu x$$

$$0 = F'(0) = c_2 \nu \Rightarrow c_2 = 0$$

$$0 = F'(\pi) = -c_1 \nu \sin \nu \pi \Rightarrow \sin \nu \pi = 0 \Rightarrow \nu \pi = n\pi$$

$$\boxed{\nu_n = n \text{ for } n=1, 2, \dots}$$

$$F_n(x) = \cos nx$$

$$\boxed{\nu_n = n \text{ for } n=0, 1, 2, \dots}$$

$$F_n(x) = \cos nx$$

$$\nu_0 = 0$$

$$F_0 = 1$$

$$\boxed{k_n = -n^2 \text{ for } n = 0, 1, \dots} \quad B_0 = \frac{2}{\pi} \quad B_{2k+1} = 0$$

$$F_n(x) = \cos nx \quad B_{2k} = -\frac{4}{\pi} \frac{1}{4k^2 - 1}$$

$$-(2n)^2 t$$

$$\dot{G}_n + (2n)^2 G_n = 0$$

$$G_n(t) = e$$

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \sum_{n=0}^{\infty} B_n \cos nx e^{-(2n)^2 t}$$

$$f(x) = u(x, 0) = \sum_{n=0}^{\infty} B_n \cos nx = B_0 + \sum_{n=1}^{\infty} B_n \cos nx$$

~~$\sin x$~~ ~~$\sum B_n$~~ $\sin x = u(x, 0)$

$$= \frac{2}{\pi} + \sum_{n=1}^{\infty} \left(-\frac{4}{\pi} \right) \frac{1}{4n^2 - 1} \cos 2nx$$

$$u(x, t) = \sum_{n=0}^{\infty} \underline{B_n} \cos nx e^{-(2n)^2 t}$$

$$= \frac{2}{\pi} + \sum_{n=2k} + \left(\sum_{n=2k+1} B_n \cos nx e^{-(2n)^2 t} \right)$$

$$= \frac{2}{\pi} + \sum_{k=1}^{\infty} \underline{B_{2k}} \cos 2kx e^{-(4k)^2 t}$$

$$= \frac{2}{\pi} + \sum_{k=1}^{\infty} \left(-\frac{4}{\pi} \right) \frac{1}{4k^2 - 1} \cos 2kx e^{-(4k)^2 t}$$

#30

$$f(x) = x \text{ on } [-\pi, \pi)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} =$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx = 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left. \frac{x^3}{3} \right|_0^{\pi} = \frac{2\pi^2}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \sum_{n=1}^{\infty} \left(\frac{2}{n} \right)^2 = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$