§12.12, #4, p602

$$\begin{cases} \dfrac{\partial w}{\partial x} + x\dfrac{\partial w}{\partial t} = x \\ w(x,0) = 1 \\ \boxed{w(0,t) = 1} \end{cases}$$

$$\mathcal{L}\{w(x,t)\} = W(x,s) = \int_0^\infty e^{-st} w(x,t)\,dt$$

$$\mathcal{L}\left\{\frac{\partial w}{\partial x}\right\} = \frac{\partial}{\partial x} W(x,s)$$

$$\mathcal{L}\left\{x\frac{\partial w}{\partial t}\right\} = x\left[sW(x,s) - w(x,0)\right]$$

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$

$$\mathcal{L}\{x\} = x\mathcal{L}\{1\} = \frac{x}{s}$$

$$\frac{\partial}{\partial x} W(x,s) + x\left[sW(x,s) - 1\right] = \frac{x}{s}$$

$$\frac{\partial}{\partial x} W + sxW = x\,\frac{s+1}{s}$$

ODE w.r.t x

$$W' + sxW = \frac{s+1}{s}x$$

$p(x) = sx$, $g(x) = \frac{s+1}{s}x$

$$W' + pW = g$$

$$W(x,s) = e^{-\int sx\,dx}\left[\int e^{\int sx\,dx}\,\frac{s+1}{s}x\,dx + \underline{c}\right]$$

$$W' + pW = g$$

$$W(x,s) = e^{-\int p(x)\,dx}\left[\int \overbrace{e^{\int p(x)\,dx}}^{=\mu} g\,dx + c\right]$$

$$\underbrace{\mu W' + \mu p W}_{\Vert} = \mu g \qquad \mu = ?$$

$$\mu W = \int \mu g + c \qquad \mu = e^{\int p\,dx}$$

$$(\mu W)' = \mu' W + \mu W'$$

$$\mu' = \mu p \qquad \frac{d\mu}{\mu} = p\,dx \qquad \ln \mu = \int p\,dx$$

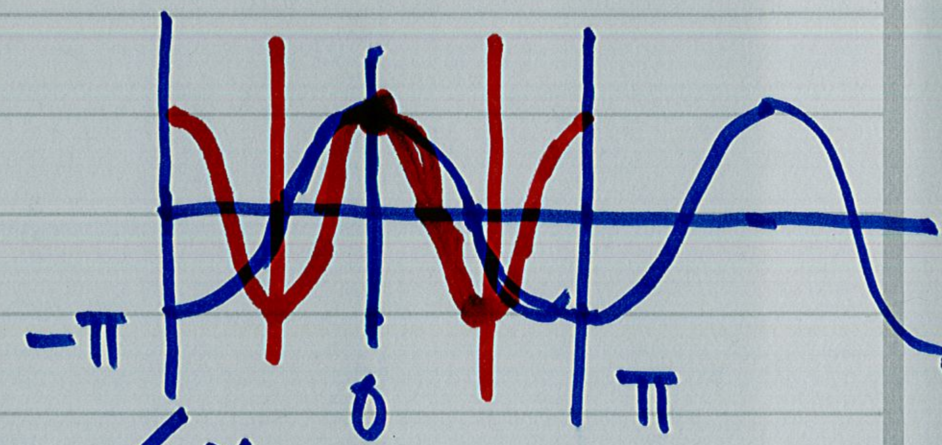
Chapter 11 Fourier Analysis

§11.1 Fourier Series

$f(x+2p)$ $\quad kp$

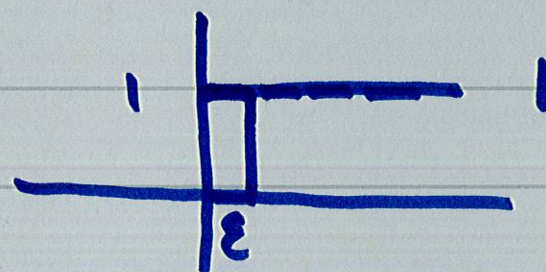
$\shortparallel$

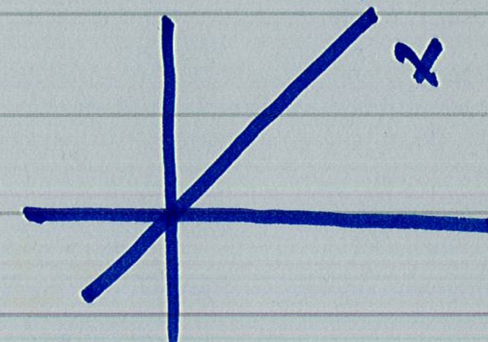
periodic function $\quad f(x+p)=f(x)\ \forall x$



examples $\quad 1$

x





2π

$\cos 2x \quad p=\pi$

$$\boxed{\cos x} = \cos(x+2\pi) \qquad p=2\pi$$

$$\cos(nx) = \cos(nx+2\pi) = \cos n\left(x+\frac{2\pi}{n}\right) \qquad \boxed{p=\frac{2\pi}{n}}$$

$$\sin x = \sin(x+2\pi) \qquad p=2\pi$$

$$\sin(nx)$$

Orthogonality

$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$, $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} \cancel{\pi}\ 1 & n=m \\ 0 & n \neq m \end{cases} = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(n+m)x + \cos(n-m)x\right] dx$$

$n \neq m$

$$= \frac{1}{2}\left[\frac{1}{n+m}\sin(n+m)x + \frac{1}{n-m}\sin(n-m)\right]_{-\pi}^{\pi} = 0$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \overset{\sin}{\cancel{\cos}} mx \, dx = \begin{cases} \cancel{\pi}\ 1 & n=m \\ 0 & n \neq m \end{cases}$$

$f(x), g(x)$ on $[-\pi, \pi]$

$$(f, g) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) g(x) \, dx$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0 = \frac{1}{2}\int_{-\pi}^{\pi} \left[\sin(n+m)x + \sin(m-n)x\right] dx$$

$$= \frac{1}{2}\left[\frac{-1}{n+m}\cos(n+m)x + -\frac{1}{m-n}\cos(m-n)x\right]_{-\pi}^{\pi}$$

$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

Trigonametric Series

$a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$ Trigonometric poly.

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$\frac{1}{2}a_0 + \sum_{n=1}^{N}\left(a_n \cos nx + b_n \sin nx\right)$$

$$\{1, \cos x, \cos 2x, \cdots, \cos Nx, \sin x, \sin 2x, \cdots, \sin Nx\}$$

Fourier Series

$$f(x + \cancel{2x}\,2\pi) = f(x) \quad \forall x \qquad p = 2\pi$$

$$\cancel{f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,dx \qquad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

Euler's formula

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f\,dx = a_0 \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} dx + \frac{1}{2\pi}\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos nx\,dx + b_n \int_{-\pi}^{\pi}\sin nx\,dx$$

$$= a_0 \cdot 1 \implies a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,dx$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$a_m \stackrel{?}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\, dx$$

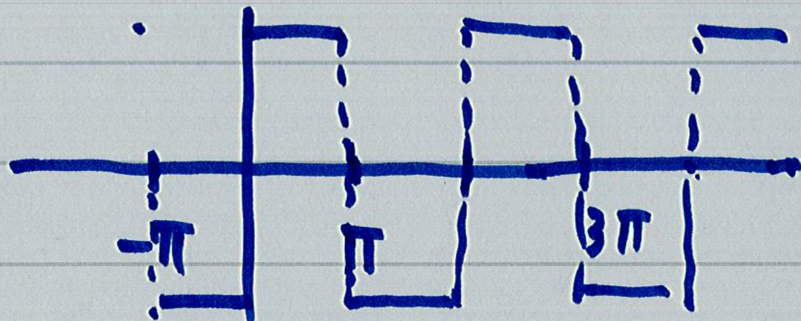
$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f \sin mx\, dx$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx = \underline{\frac{1}{\pi}\cdot\frac{1}{2}a_0\int_{-\pi}^{\pi}\cos mx\, dx} + \sum_{n=1}^{\infty}\left(\frac{a_n}{\pi}\int_{-\pi}^{\pi}\underline{\cos nx \cos mx}\, dx + \frac{b_n}{\pi}\int_{-\pi}^{\pi}\underline{\sin nx \cos mx\, dx}\right)$$

$$= a_m$$

Ex. 2 (Periodic Rectangular Wave)

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}, \quad f(x+2\pi) = f(x). \quad \text{Find F-S.}$$



$$a_0 = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\left[\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right]$$

$$= \frac{2}{\pi}\left[-k\pi + k\pi\right] = 0$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\cos nx + \int_{0}^{\pi} k\cos nx\,dx\right] = \frac{1}{\pi}\left[\frac{-k}{n}\sin nx\Big|_{-\pi}^{0} + \frac{k}{n}\sin nx\Big|_{0}^{\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\sin nx + \int_{0}^{\pi} k\sin nx\right] = \frac{1}{\pi}\left[\frac{k}{n}\cos nx\Big|_{-\pi}^{0} - \frac{k}{n}\cos nx\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{k}{n}\left(1 - \cos n\pi\right) - \frac{k}{n}\left(\cos n\pi - 1\right)\right]$$

$$b_n = \frac{2}{\pi} \cdot \frac{k}{n} \left[1 - \cos n\pi \right]$$

$$= \frac{2k}{n\pi} \left[1 - (-1)^{n} \right]$$

$$= 2k$$

$\cos \pi = -1$

$\cos 2\pi = 1$

$\cos 3\pi = -1$

$1-(-1)^n$	
$= 2$	$n=1$
0	$n=2$
2	$n=3$
0	$n=4$

(b) $\cancel{b_{2n} = 0}$ $\cancel{b_{2n+1}}$

$$b_{2l} = 0$$

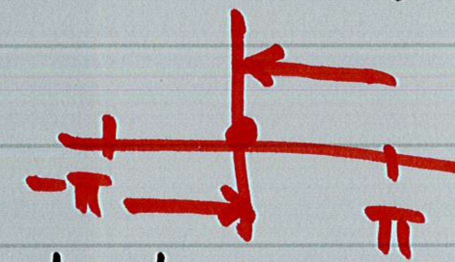
$$b_{2l+1} = \frac{4k}{(2l+1)\pi}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{l=0}^{\infty} \frac{4k}{(2l+1)\pi} \sin (2l+1)\pi$$

Thrm 2 Assumptions:

(1) $f(x+2\pi)=f(x)\ \forall x$; (2) f is ~~cont~~ piecewise continuous in $[-\pi,\pi]$;

(3) at discont. pt x_0, $f'(x_0^-)$ and $f'(x_0^+)$ exist.



$$\Longrightarrow \frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)=\begin{cases} f(x) & \text{at cont. pt.} \\ \dfrac{f(x^-)+f(x^+)}{2} & \text{at discont. pt.} \end{cases}$$