

§11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansion.

• $f(x+2L) = f(x) \quad \forall x \iff f$ has period $p=2L$

$$v = \frac{\pi}{L}x \quad \Rightarrow \quad f\left(\frac{L}{\pi}v\right) \text{ is a periodic function with } p=2\pi$$
$$x = \frac{L}{\pi}v$$

$$f(x) = f\left(\frac{L}{\pi}v\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nv + b_n \sin nv \right)$$
$$= \frac{1}{2}a_0 + \sum \left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) \cos nv \, dv \stackrel{x = \frac{L}{\pi}v}{=} \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi}{L}x \, dx, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi}{L}x \, dx$$

Ex. 1

$$f(x) = \begin{cases} 0 & x \in (-2, -1) \\ k & x \in (-1, 1) \\ 0 & x \in (1, 2) \end{cases}$$

$$p = 2L = 4$$

Ex. 3

$$u(t) = \begin{cases} 0 & t \in (-L, 0) \\ E \sin \omega t & t \in (0, L) \end{cases}$$

$$p = 2L = \frac{2\pi}{\omega}$$

• Even and Odd Functions

even function $f(-x) = f(x) \quad \forall x$

e.g., $1, x^2, x^4, \dots, \cos nx$

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

odd function $f(-x) = -f(x) \quad \forall x$

e.g., $x, x^3, \dots, \sin nx$

$$\int_{-a}^a f(x) dx = 0$$

cosine series $f(-x) = f(x)$ and $f(x+2L) = f(x) \quad \forall x$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L}x$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L}x dx$$

sine series $f(-x) = -f(x)$ and $f(x+2L) = f(x) \quad \forall x$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}x$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L}x dx$$

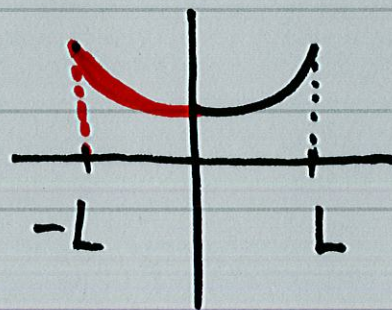
Ex. 5 $f(x) = x + \pi, x \in [-\pi, \pi], f(x+2\pi) = f(x)$
 $= \underbrace{f_1(x)}_{\text{odd}} + f_2(x)$ with $f_1(x) = x$ and $f_2(x) = \pi$

Half-Range Expansions

$f(x)$ is defined on $(0, L)$ and $f(x+2L) = f(x)$

even extension

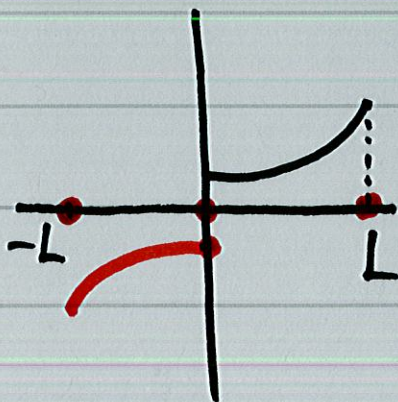
$$f(x) = \begin{cases} f(-x) & x \in (-L, 0) \\ f(x) & x \in (0, L) \end{cases}$$



cosine series

odd extension

$$f(x) = \begin{cases} -f(-x) & x \in (-L, 0) \\ 0 & x = 0, \pm L \\ f(x) & x \in (0, L) \end{cases}$$



sine series

Ex. 6

$$f(x) = \begin{cases} \frac{2k}{L}x & x \in (0, \frac{L}{2}) \\ \frac{2k}{L}(L-x) & x \in (\frac{L}{2}, L) \end{cases}$$

Find cosine and sine series.