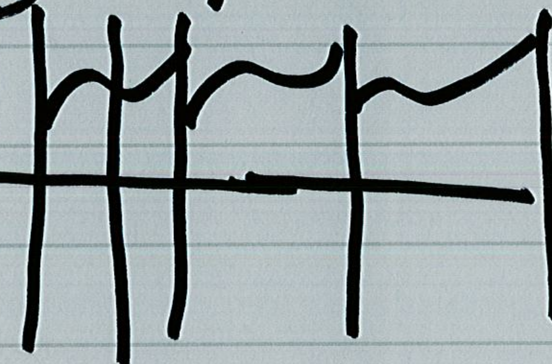
Fourier Series

$f(x)$ on $[-\pi, \pi]$ $\quad f(x+2\pi) = f(x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$



$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \,dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \,dx$$

$f(x)$ is p. cont. on $[-\pi, \pi]$

§11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansion.

- $f(x+2L) = f(x) \quad \forall x \iff f$ has period $p = 2L$

$= f\left(x + \frac{L}{\pi} 2\pi\right) = f\left(\frac{L}{\pi}\left[\frac{\pi}{L}x + 2\pi\right]\right) = f\left(\frac{L}{\pi}[v + 2\pi]\right)$

$v = \frac{\pi}{L}x$

$x = \frac{L}{\pi}v$

$\Rightarrow f\left(\frac{L}{\pi}v\right)$ is a periodic function with $p = 2\pi$

$f\left(\frac{L}{\pi}(v+2\pi)\right) = f(x+2L) = f(x) = f\left(\frac{L}{\pi}v\right)$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$f(x) = f\left(\frac{L}{\pi}v\right) = \cancel{\frac{1}{2}}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nv + b_n \sin nv\right)$$

$$\boxed{f(x) = \cancel{\frac{1}{2}}a_0 + \sum\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)}$$

$-L \to L$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\cos nv\,dv \overset{x=\frac{L}{\pi}v}{=\!=\!=} \boxed{\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx}, \quad \boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx}$$

$dx = \frac{L}{\pi}dv$

Ex. 1

$$f(x)=\begin{cases} 0 & x\in(-2,-1) \\ k & x\in(-1,1) \\ 0 & x\in(1,2) \end{cases} = \frac{k}{2}+\frac{2k}{\pi}\sum_{l=0}^{\infty}(-1)^{l}\frac{1}{2l+1}$$

$p=2L=4 \Longrightarrow L=2$

$n=1,\ 2,\ 3,\ 4$

$1,\ 0,\ -1,\ 0$

$1,\ 0,\ -1,\ 0$

$2l$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)\,dx=\frac{1}{4}\int_{-1}^{1}k\,dx=\frac{k}{2}$$

$$a_n=\frac{1}{2}\int_{-2}^{2}f(x)\cos\frac{n\pi}{2}x\,dx=\frac{k}{2}\int_{-1}^{1}\cos\frac{n\pi}{2}x\,dx=\frac{k}{2}\left[\cancel{\frac{2}{n\pi}}\sin\frac{n\pi x}{2}\right]_{-1}^{1}$$

$\cancel{\frac{2}{n\pi}}$

$$=\frac{k}{2}\cdot\frac{2}{n\pi}\left[\sin\frac{n\pi}{2}+\sin\frac{n\pi}{2}\right]=\frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

$\sum\frac{1}{n}$

$$\underbrace{a_{2l+1}}_{n}=\frac{2k}{(2l+1)\pi}(-1)^{l} \qquad l=0,1,2,\dots$$

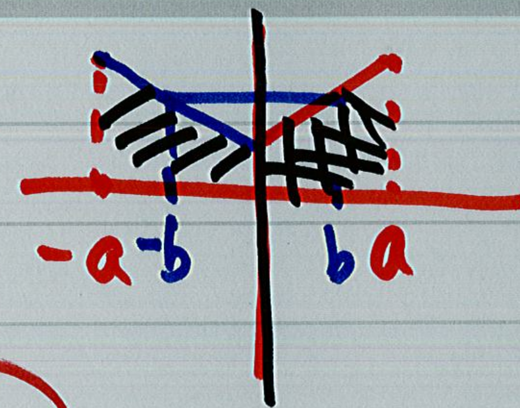
$$b_n=\frac{k}{2}\int_{-1}^{1}\sin\frac{n\pi}{2}x\,dx=\frac{k}{2}\left[-\frac{2}{n\pi}\cos\frac{n\pi}{2}x\right]_{-1}^{1}=-\frac{k}{n\pi}\left[\cos\frac{n\pi}{2}-\cos\frac{n\pi}{2}\right]=0$$

Ex. 3

$$u(t)=\begin{cases} 0 & t\in(-L,0) \\ E\sin\omega t & t\in(0,L) \end{cases} \qquad p=2L=\frac{2\pi}{\omega}$$

• Even and Odd Functions

even function $\quad f(-x) = f(x) \quad \forall x$



sym w.r.t. $x=0$

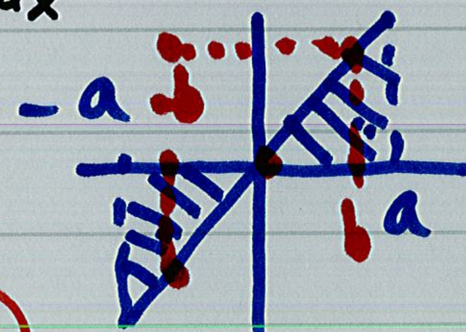
e.g., $1, x^2, x^4, \cdots, \cos nx$

$y=-x$

$$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

odd function $\quad f(-x) = -f(x) \quad \forall x$



sym w.r.t. $(0,0)$

e.g., $x, x^3, \cdots, \sin nx$

$$\int_{-a}^{a} f(x)\,dx = 0$$

cosine series $\quad f(-x) = f(x)$ and $f(x+2L) = f(x) \quad \forall x$

$$f(x) = \cancel{\tfrac{1}{2}}\, a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

$f(x)g(x)$: $\quad f(-x)g(-x) = f(x)\left[-g(x)\right] = -f(x)g(x)$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi}{L} x \, dx, \quad 0 = b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx$$

sine series $\quad f(-x) = -f(x)$ and $f(x+2L) = f(x) \quad \forall x$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f \cos \frac{n\pi}{L} x \, dx = 0$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$$

Ex. 5 $f(x) = x + \pi, \quad x \in [-\pi, \pi], \quad f(x+2\pi) = f(x)$

$$= \underbrace{f_1(x)}_{\text{odd}} + f_2(x) \quad \text{with } f_1(x) = x \text{ and } f_2(x) = \pi$$

$n = 1, 2, \cdots$

$u = x, \quad v' = \sin nx$

$u' = 1, \quad v = -\frac{1}{n}\cos nx$

$$f_1(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[-\frac{x}{n}\cos nx \Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx \right] dx$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos n\pi + \frac{1}{n}\,\frac{1}{n}\sin nx \Big|_0^{\pi} \right]$$

$$= -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

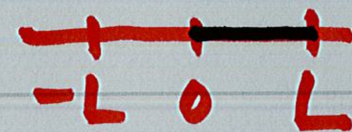
$$f_1(x) = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin nx$$

$$f_2(x) = \pi$$

$$f(x) = \pi + 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin nx$$

$$\frac{1}{n^2}\cos nx$$

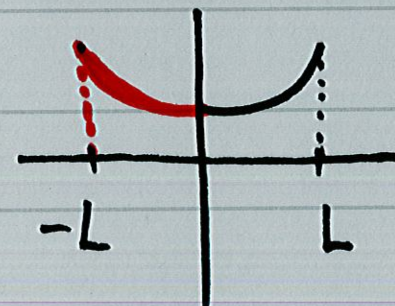
Half-Range Expansions



$f(x)$ is defined on $(0, L)$ and $f(x+2L) = f(x)$

even extension

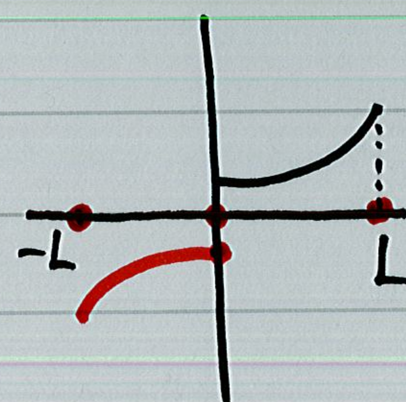
$$f(x) = \begin{cases} f(-x) & x \in (-L, 0) \\ f(x) & x \in (0, L) \end{cases}$$



cosine series

odd extension

$$f(x) = \begin{cases} -f(-x) & x \in (-L, 0) \\ 0 & x = 0, \pm L \\ f(x) & x \in (0, L) \end{cases}$$



sine series

Ex. 6

$$f(x) = \begin{cases} \frac{2k}{L} x & x \in \left(0, \frac{L}{2}\right) \\ \frac{2k}{L}(L-x) & x \in \left(\frac{L}{2}, 0\right) \end{cases}$$

Find cosine and sine series.