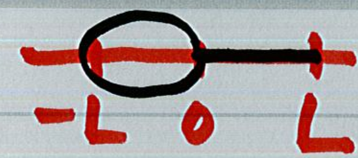
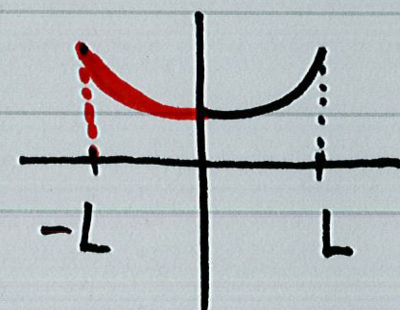
Half-Range Expansions



$f(x)$ is defined on $(0, L)$ and $f(x+2L) = f(x)$

even extension

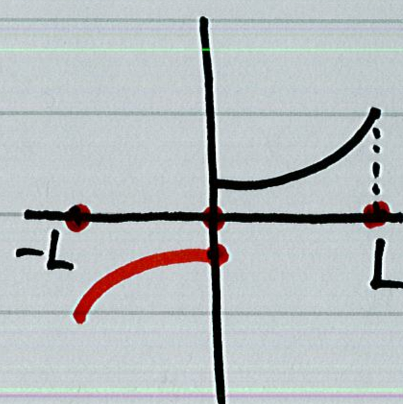
$$f(x) = \begin{cases} f(-x) & x \in (-L, 0) \\ f(x) & x \in (0, L) \end{cases}$$



cosine series

odd extension

$$f(x) = \begin{cases} -f(-x) & x \in (-L, 0) \\ 0 & x = 0, \pm L \\ f(x) & x \in (0, L) \end{cases}$$

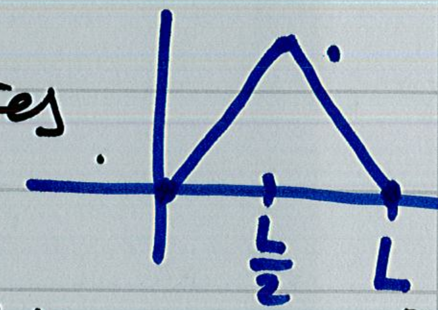


sine series

Ex. 6

$$f(x)=\begin{cases}\frac{2k}{L}x & x\in\left(0,\frac{L}{2}\right)\\ \frac{2k}{L}(L-x) & x\in\left(\frac{L}{2},L\right)\end{cases}$$

Find cosine and sine series



$$a_0=\frac{1}{L}\int_0^L f\,dx=\frac{1}{L}\left[\int_0^{\frac{L}{2}}+\int_{\frac{L}{2}}^{L}f\,dx\right]=\frac{k}{2},$$

$$a_n=\frac{2}{L}\left[\int_0^{\frac{L}{2}}\frac{2k}{L}x\cos\frac{n\pi}{L}x\,dx+\int_{\frac{L}{2}}^{L}\frac{2k}{L}(L-x)\cos\frac{n\pi x}{L}dx\right]$$

$$=\frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2}-\cos n\pi-1\right)$$

$$f(x)=\frac{k}{2}+\frac{4k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(2\cos\frac{n\pi}{2}-\cos n\pi-1\right)\cos\frac{n\pi}{L}x$$

$$a_{2(2l+1)}=-\frac{4^2k}{4(2l+1)^2\pi^2}=\frac{-4k}{\pi^2(2l+1)^2}$$

$$a_{2(2l)}=a_{2l+1}=0$$

$$b_n=\frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}=(-1)^{n+1}\frac{8k}{n^2\pi^2}$$

$$f(x)=\frac{8k}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}.$$

§11.3 Solving ODEs with Fourier Series

$$y=\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

Ex. 1 $y'+2y=f(x)$ ~ $\boxed{f(x+2\pi)=f(x)}$ $\forall x$, 1, $\cos x$, ⋯, $\cos nx$, ⋯ $\sin x$, ⋯, $\sin nx$, ⋯

Set $f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$

$y(x)=c_0+\sum_{n=1}^{\infty}\left(c_n\cos nx+d_n\sin nx\right)$ $\quad c_n, d_n$ — to be determined $=?$

$$y'=\sum_{n=1}^{\infty}\cancel{\frac{c_n}{n}\sin nx}\left[-nc_n\sin nx+nd_n\cos nx\right]$$

$$y'+2y=2c_0+\sum_{n=1}^{\infty}\left[(nd_n+2c_n)\cos nx+(-nc_n+2d_n)\sin nx\right]$$

$$\therefore\ =a_0+\sum_{n=1}^{\infty}\left[a_n\cos nx+b_n\sin nx\right]$$

$2c_0=a_0$, $\quad nd_n+2c_n=a_n$, $\quad -nc_n+2d_n=b_n$

$c_0=\frac{1}{2}a_0$

$$\begin{bmatrix}-n & 2\\ 2 & n\end{bmatrix}\begin{bmatrix}c_n\\ d_n\end{bmatrix}=\begin{bmatrix}b_n\\ a_n\end{bmatrix}$$

$$\begin{bmatrix}c_n\\ d_n\end{bmatrix}=-\frac{1}{n^2+4}\begin{bmatrix}n & -2\\ -2 & -n\end{bmatrix}\begin{bmatrix}b_n\\ a_n\end{bmatrix}=-\frac{1}{n^2+4}\begin{bmatrix}nb_n-2a_n\\ -2b_n-na_n\end{bmatrix}$$

Ex. 2 $\quad y'' + 2y' + 2y = f(t), \quad \underline{f(t+2\pi) = f(t)}$

$$y(t) = y_h(t) + y_p(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + y_p(t)$$

$$y_h'' + 2y_h' + 2y_h = 0 \quad 0 = s^2 + 2s + 2 = (s+1)^2 + 1 \quad s = -1 \pm i \quad e^{-t}\cos t,\ e^{-t}\sin t$$

$$y_p'' + 2y_p' + 2y_p = f = a_0 + \sum\left(a_n \cos nx + b_n \sin nx\right)$$

$$= 2c_0 + \sum\Big(\left(-n^2c_n + 2nd_n + 2c_n\right)\cos nx + \left(-n^2d_n - 2nc_n + 2d_n\right)\sin nx\Big)$$

$$y_p = c_0 + \sum\left(c_n \cos nx + d_n \sin nx\right)$$

$$y_p' = \sum\left(-nc_n \sin nx + nd_n \cos nx\right)$$

$$y_p'' = \sum\left(-n^2c_n \cos nx - n^2 d_n \sin nx\right)$$

$$c_0 = \frac{a_0}{2}, \quad \begin{cases} (2-n^2)c_n + 2nd_n = a_n \\ -2nc_n + (2-n^2)d_n = b_n \end{cases}$$

§11.4 Approximation by Trigonometric Polynomials

$f(x)$ is defined on $(-\pi, \pi)$, $\underline{f(x+2\pi) = f(x)}$

$$f(x) \approx F(x) = \text{[redacted]}\, a_0 + \sum_{n=1}^{N} \left(a_n \cos nx + b_n \sin nx \right)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

general trigonometric poly.

$$G(x) = \text{[redacted]}\, \underline{A_0} + \sum_{n=1}^{N} \left(\underline{A_n} \cos nx + \underline{B_n} \sin nx \right)$$

question

$$\min_{G} \| f - G \|^2 = ? \quad \| f(x) - \underline{F(x)} \|^2$$

where $\| f \|^2 = \int_{-\pi}^{\pi} f^2\,dx$ $\quad$ L^2-norm $= (f, f)$

Thrm

$$\boxed{\min_{G} \|f-G\|^2 = \|f-F\|^2} = \underline{\|f\|^2 - \|F\|^2}$$

$$\|f-F\|^2 = \|(f-G)+(G-F)\|^2 = ((f-G)+(G-F), (f-G)+(G-F))$$

$$= \|f-G\|^2 + \|G-F\|^2 + 2\underbrace{(f-G, G-F)}$$

$$f-G = f-F + \underline{F-G}$$

$$\|f-F\|^2 = \|f-G\|^2 - \|G-F\|^2$$

$$\underbrace{(f-G, G-F)} = 2\boxed{(f-F, G-F)} - 2(G-F, G-F)$$

$$\cos(n+1)x \rightarrow (f-F, G-F) = 0$$

$$-2\|F-G\|^2$$

$$\leq \|f-G\|^2$$

Bessel's Inequality

$$\frac{1}{\pi}\|f\|^2 \geq \frac{1}{\pi}\|F\|^2 = 2a_0^2 + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)$$

Parseval's Identity

$$\frac{1}{\pi}\|f\|^2 = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

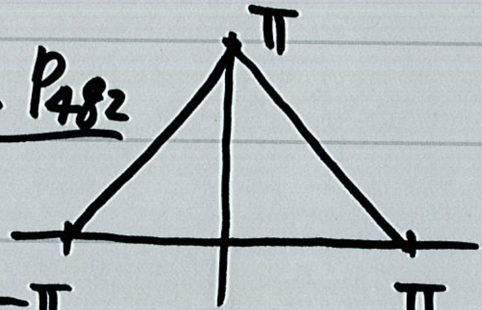
#13, P498, §11.4

$$\sum_{l=0}^{\infty}\frac{1}{(2l+1)^4} = 1+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$$

Prove $1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots = \boxed{\frac{\pi^4}{96}}$

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx = 2a_0^2+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)}$$

#17. P482



F-series $f(x)=\frac{\pi}{2}+\sum_{l=0}^{\infty}\frac{4}{\pi}\frac{1}{(2l+1)^2}\cos(2l+1)x$

$$f(x)=\begin{cases}\pi-x & x\in(0,\pi)\\ \pi+x & x\in(-\pi,0)\end{cases}$$ even

$$LHS=\frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi-x)^2\,dx = \frac{2}{\pi}\cdot\frac{-1}{3}\cdot(\pi-x)^3\Big|_0^{\pi} = \frac{2\pi^2}{3}$$

$$RHS=\frac{\pi^2}{2}+\sum\frac{16}{\pi^2}\cdot\left(\frac{1}{(2l+1)^2}\right)^2$$

$$=\frac{\pi^2}{2}+\frac{16}{\pi^2}\sum_{l=0}^{\infty}\frac{1}{(2l+1)^4}$$

$b_n=0$

$a_0=?,\ a_n=?$

$a_{2l}=0$

$$\left(\frac{2}{3}-\frac{1}{2}\right)\pi^2\cdot\frac{\pi^2}{16}=\sum_{l=0}^{\infty}\frac{1}{(2l+1)^4}$$