§11.5 Sturm-Liouville Problems. Orthogonal Functions

trigonometric system $\Longrightarrow$ orthogonal system

Sturm-Liouvill Problem

$$\begin{cases} \left(p(x)\,y'\right)' + \left(q(x) + \lambda\, r(x)\right) y = 0 \qquad x \in [a,b] \\ \left[k_1 y + k_2 y'\right](a) = 0 \\ \left[l_1 y + l_2 y'\right](b) = 0 \end{cases}$$

BCs

eigenfunction $\quad y(x) \not\equiv 0$

eigenvalue $\quad \lambda$

Ex. 1 $\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0, \ y(\pi) = 0 \end{cases}$

$\underline{\lambda = -\nu^2 < 0}$

$\underline{\lambda = 0}$

$\underline{\lambda = \nu^2 > 0}$ $\lambda = \nu^2$ with $\nu = 1, 2, 3, \cdots$; $y(x) = \sin(\nu x)$

$(\nu > 0)$

Orthogonal Functions

- weight function $\quad r(x) > 0$
- inner product $\quad (u, v) = \int_a^b r(x)\, u(x)\, v(x)\, dx$
- norm $\quad \|u\| = (u, u)^{\frac{1}{2}} = \sqrt{\int_a^b r u^2 dx}$
- orthogonal $\quad u \perp v \iff (u, v) = 0$
- orthonormal system $\quad \{y_1, y_2, \cdots, y_n, \cdots\}$

$$\iff \quad (y_n, y_m) = \underbrace{\delta_{mn}}_{\text{Kronecker symbol}} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Ex. 2 $\left\{ \sin x/\sqrt{\pi},\ \sin 2x/\sqrt{\pi},\ \cdots \right\}$ w.r.t $r=1$ and $[a,b]=[-\pi,\pi]$

Thrm 1 (Orthogonality of Eigenfunctions of S-L Prob.)

Assumptions (1) p, q, r, p' — real-valued, cont. on $[a,b]$

(2) $r>0$ on $[a,b]$

(3) λ_m — eigenvalue, y_m — eigenfunction

$$\Longrightarrow \quad (y_m, y_n) = \int_a^b r(x)\, y_m(x)\, y_n(x)\, dx = 0 \qquad (m \neq n)$$

BCs

(1) $p(a)=0 \Longrightarrow [k_1 y + k_2 y'](a)=0$ can be dropped

$p(b)=0 \Longrightarrow [l_1 y + l_2 y'](b)=0$ can be dropped

$\Big\}$ singular problem

(2) $p(a)=p(b) \Longrightarrow$ periodic BCs :

$$\begin{cases} y(a)=y(b) \\ y'(a)=y'(b) \end{cases}$$

Ex. 4 Legendre Polynomials

$$(1-x^2)y'' - 2xy' + n(n+1)y = \left[(1-x^2)y'\right]' + \lambda y = 0 \qquad \lambda = n(n+1)$$

$p(-1) = p(1) = 0 \implies$ no BCs needed.

singular S-L prob on $[-1, 1]$

<u>eigenvalue</u> $\lambda = 0,\ 1\cdot 2,\ 2\cdot 3,\ 3\cdot 4,\ \cdots$

<u>eigenfunction</u> $P_0,\ P_1,\ P_2,\ P_3,\ \cdots$

#9, P503, §11.5

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0, \; y'(L) = 0 \end{cases}$$