§11.5 Sturm-Liouville Problems. Orthogonal Functions

trigonometric system $\Longrightarrow$ orthogonal system

Sturm-Liouvill Problem

$$\begin{cases} \left(p(x)\,y'\right)' + \left(q(x) + \lambda\, r(x)\right) y = 0 & x \in [a, b] \\ \left[k_1 y + k_2 y'\right](a) = 0 & \\ \left[l_1 y + l_2 y'\right](b) = 0 & \text{BCs} \end{cases}$$

eigenfunction $y(x) \not\equiv 0$

eigenvalue λ

Orthogonal Functions

- weight function $\quad r(x) > 0$
- inner product $\quad (u, v) = \int_a^b r(x)\, u(x)\, v(x)\, dx$
- norm $\quad \|u\| = (u, u)^{\frac{1}{2}} = \sqrt{\int_a^b r u^2 dx}$
- orthogonal $\quad u(x) \perp v(x) \Longleftrightarrow (u, v) = 0$
- orthonormal system $\quad \{ y_1, y_2, \cdots, y_n, \cdots \}$

$(y_n, y_m) = 0 \quad \|y_n\| = 1$
$(n \neq m)$

$$\Longleftrightarrow \quad (y_n, y_m) = \underbrace{\delta_{mn}}_{\text{Kronecker symbol}} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

$\perp$

$0 = (\vec{x}, \vec{y}) = \sum_{i=1}^{n} x_i y_i$

$\|\vec{x}\| = \sqrt{(\vec{x}, \vec{x})}$

Ex. 1

$$\begin{cases} y'' + \lambda y = 0, & [0,\pi] \quad p=1,\ q=0,\ r=1 \\ y(0)=0,\ y(\pi)=0 & k_1=1,\ k_2=0,\ l_1=1,\ l_2=0 \end{cases}$$

$\lambda = ?\quad y(x)\ ?$

$A\vec{x} = \lambda\vec{x},\ \lambda,\ \vec{x}$

$\underline{\lambda = -\nu^2 < 0}$ $(\nu > 0)$

$\boxed{0 = s^2 + \lambda = s^2 - \nu^2} \Rightarrow s^2 = \nu^2 = s = \pm\sqrt{\nu^2} = \pm\nu$

$y(x) = c_1 e^{-\nu x} + c_2 e^{\nu x}$

$\equiv 0$

$0 = y(0) = c_1 + c_2$

$0 = y(\pi) = c_1 e^{-\nu\pi} + c_2 e^{\nu\pi}$

$\begin{bmatrix} 1 & 1 \\ e^{-\nu\pi} & e^{\nu\pi} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$|A| = e^{\nu\pi} - e^{-\nu\pi} \neq 0$

$c_1 = c_2 = 0$

$\underline{\lambda = 0}$

$y'' = 0 \quad y = c_1 + c_2 x$

$0 = y(0) = c_1$

$0 = y(\pi) = c_2\pi \Rightarrow c_2 = 0 \Longrightarrow y(x) \equiv 0$

$\underline{\lambda = \nu^2 > 0}$ $(\nu > 0)$

$\lambda = \nu^2$ with $\nu = 1, 2, 3, \cdots;\quad y(x) = \sin(\nu x)$

$0 = s^2 + \lambda = s^2 + \nu^2 \Rightarrow s = \pm\sqrt{-\nu^2} = \pm\nu i \qquad e^{\nu i x} = (\cos\nu x + i\sin\nu x)$

$y(x) = c_1\cos\nu x + c_2\sin\nu x$

$0 = y(0) = c_1$

$0 = y(\pi) = c_2 \underline{\sin\nu\pi}$

$\underline{\sin\nu\pi = 0} \qquad \underline{\nu = 1, 2, 3, \cdots, n, \cdots}$

$\lambda = \nu^2 = 1, 4, 9, \cdots, n^2, \cdots$ eigenvalues

$y(x) = \sin\nu x = \sin x, \sin 2x, \cdots, \sin nx, \cdots,$ eigenfunctions

$$\lambda = 1, \quad 4, \quad 9, \quad \cdots, \quad n^2, \quad \cdots \qquad \text{eigenvalues}$$

$$y = \underline{\sin x, \ \sin 2x, \ \sin 3x, \ \cdots, \ \sin nx, \ \cdots} \qquad \text{eigenfunctions}$$

$$\int_{-\pi}^{\pi} \frac{\sin nx}{\sqrt{\pi}} \, \frac{\sin mx}{\sqrt{\pi}} \, dx = \begin{cases} 0 & n \neq m \\ \cancel{\pi} \ 1 & n = m \end{cases}$$

$$\left\{ \frac{\sin x}{\sqrt{\pi}}, \ \frac{\sin 2x}{\sqrt{\pi}}, \ \cdots, \ \frac{\sin nx}{\sqrt{\pi}}, \ \cdots \right\}$$

<u>orthonormal</u> system

Ex. 2 $\left\{ \sin x/\sqrt{\pi}, \ \sin 2x/\sqrt{\pi}, \ \cdots \right\}$ w.r.t $r=1$ and $[a,b]=[-\pi,\pi]$

Thrm 1 (Orthogonality of Eigenfunctions of S-L Prob.)

Assumptions (1) p, q, r, p' — real-valued, cont. on $[a,b]$

(2) $r>0$ on $[a,b]$

(3) λ_m — eigenvalue, y_m — eigenfunction

$$\Longrightarrow \quad (y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x)\, dx = 0 \qquad (m \neq n)$$

BCs

(1) $p(a)=0 \Longrightarrow [k_1 y + k_2 y'](a)=0$ can be dropped

$p(b)=0 \Longrightarrow [l_1 y + l_2 y'](b)=0$ can be dropped

$\Big\}$ singular problem

(2) $p(a)=p(b) \Longrightarrow$ periodic BCs: $\begin{cases} y(a)=y(b) \\ y'(a)=y'(b) \end{cases}$

Ex. 4 Legendre Polynomials

$p = 1 - x^2,\ q = 0,\ r = 1$

$[-1, 1]$

$$(1-x^2)y'' - 2xy' + n(n+1)y = \left[(1-x^2)y'\right]' + \lambda y = 0 \qquad \lambda = n(n+1)$$

$p(-1) = p(1) = 0 \Longrightarrow$ no BCs needed.

singular S-L prob on $[-1, 1]$

eigenvalue $\quad \lambda = 0,\ 1\cdot 2,\ 2\cdot 3,\ 3\cdot 4,\ \cdots$

eigenfunction $\quad P_0,\ P_1,\ P_2,\ P_3,\ \cdots$

$$P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{3}{2}x^2 - \frac{1}{2}$$

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

$$P_n(x) = \frac{1}{2^n n!}\left[(x^2-1)^n\right]^{(n)}$$ Rodrigues' Formula

#9, P503, §11.5

$$\begin{cases} y''+\lambda y=0 \text{ on } [0,L] \\ y(0)=0,\ y'(L)=0 \end{cases}$$

• $\underline{\lambda=-\nu^2<0 \quad (\nu>0)}$

$0=s^2+\lambda=s^2-\nu^2 \Longrightarrow s=\pm\nu,\quad e^{-\nu x},\quad e^{\nu x}$

$y(x)=\underline{c_1}e^{-\nu x}+\underline{c_2}e^{\nu x},\quad 0=y(0)=c_1+c_2$

$y'(x)=-\nu c_1 e^{-\nu x}+\nu c_2 e^{\nu x},\quad 0=y'(L)=\cancel{\nu}\left[-c_1 e^{-\nu L}+c_2 e^{-\nu L}\right]$

$$\begin{bmatrix} 1 & 1 \\ -e^{-\nu L} & e^{\nu L} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad |A|=e^{\nu L}+e^{-\nu L}\neq 0$$

$\Rightarrow c_1 = c_2 = 0 \Rightarrow y(x)\equiv 0$

• $\underline{\lambda=0}$ $\quad y''=0,\quad y'=c_2,\quad 0=y'(L)=c_2 \Longrightarrow y'(x)=0$

$y(x)=c_1,\quad 0=y(0)=c_1 \Longrightarrow y(x)\equiv 0$

• $\underline{\lambda=\nu^2 \quad (\nu>0)}$ $\quad 0=s^2+\lambda=s^2+\nu^2 \Longrightarrow s=\pm\nu i$

$y(x)=c_1\cos\nu x+c_2\sin\nu x,\quad 0=y(0)=c_1$

$y'(x)=-\nu c_1\sin\nu x+\nu c_2\cos\nu x,\quad 0=y'(L)=\cancel{\nu}\,\cancel{c_2}^{\neq 0}\cos\nu L$

$\nu L=\frac{\pi}{2},\ \frac{3\pi}{2},\ \dots,\ \frac{2l+1}{2}\pi,\ \dots$

$\nu=\frac{\pi}{2L},\ \frac{3\pi}{2L},\ \dots,\ \frac{2l+1}{2L}\pi,\ \dots$

$$\nu = \frac{\pi}{2L}, \frac{3\pi}{2L}, \cdots, \frac{2l+1}{2L}\pi, \cdots, \qquad l = 0, 1, 2, \cdots$$

$$\lambda = \left(\frac{\pi}{2L}\right)^2, \left(\frac{3\pi}{2L}\right)^2, \cdots, \left(\frac{2l+1}{2L}\pi\right)^2, \cdots$$

$$y(x) = \sin\nu x = \sin\frac{\pi x}{2L}, \sin\frac{3\pi x}{2L}, \cdots, \sin\frac{(2l+1)\pi x}{2L}, \cdots.$$

$[0, L]$

$r = 1$