§11.6 Orthogonal Series. Generalized Fourier Series.

orthogonal system on $[a, b]$ w.r.t. $r(x) > 0$ $\quad \{ y_0(x), y_1(x), \cdots, y_n(x), \cdots \}$

orthogonal series/expansion or generalized F-series

$$f(x) = \sum_{m=1}^{\infty} a_m y_m(x), \qquad a_m = \frac{(f, y_m)}{\|y_m\|}$$

Ex.1 Fourier-Legendre Series

Legendre Polynomials $\{P_0(x), P_1(x), \cdots, P_n(x), \cdots\}$

$$P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{3}{2}x^2 - \frac{1}{2}, \quad \cdots, \quad P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

3-term recursive relation

orthogonality $[-1, 1]$, $r(x) = 1$

$$(P_n, P_m) = \int_{-1}^{1} P_n(x) P_m(x)\, dx = \begin{cases} \dfrac{2}{2m+1} & n = m \\ 0 & n \neq m \end{cases}$$

Fourier-Legendre Series

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x)$$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) \, dx$$

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$$(x+1)^2 =$$

Mean Square Convergence. Completeness.

- $\overline{\lim_{k\to\infty}} f_k(x) = f(x) \iff \overline{\lim_{k\to\infty}} \| f_k - f \| = 0$

- $\sum_{m=1}^{\infty} a_m y_m(x)$ converges to $f(x) \iff \overline{\lim_{k\to\infty}} s_k(x) = f(x)$ where $s_k(x) = \sum_{m=1}^{k} a_m y_m(x)$

Let S be a set of functions defined on $[a, b]$
and $\{ y_0, y_1, \cdots, y_m, \cdots \}$ be an orthonormal set

- $\{ y_0, y_1, \cdots \}$ is complete in $S \iff \forall f \in S,\ \exists \sum_{m=0}^{\infty} a_m y_m(x)$

  s.t. $f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$

Parseval Equality

$$\|f\|^2 = \sum_{m=0}^{\infty} a_m^2$$

Bessel's Inequality

$$\|f\|^2 \geq \sum_{m=0}^{k} a_m^2$$

Thrm (Completeness)

$$(f, y_m) = 0 \ \forall m \Longrightarrow \|f\| = 0 \xrightarrow{f \text{ is cont.}} f(x) \equiv 0$$