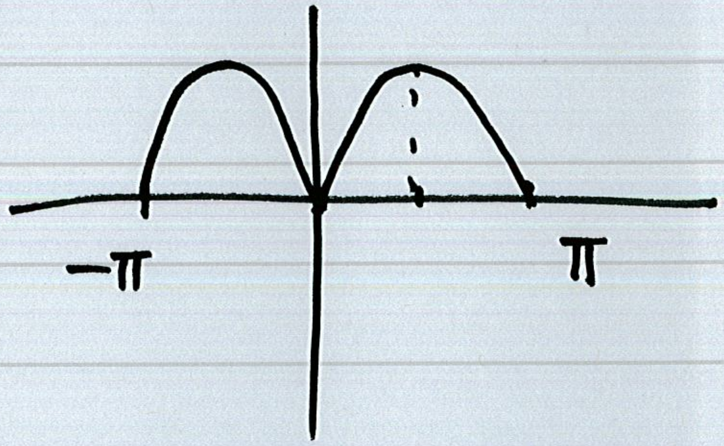
#29 on p491, §11.2   F-cosine, F-sine

$f(x) = \sin x$ on $[0, \pi]$



$L = 2\pi$

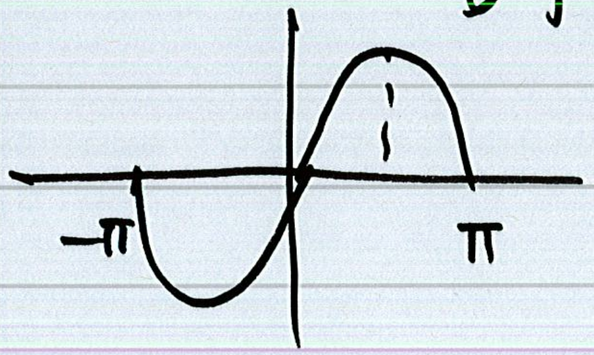
F-cosine

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx\,dx$$

F-sine

$f(x) = \sin x$ on $[-\pi, \pi]$



$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx\,dx$$

§11.6 Orthogonal Series. Generalized Fourier Series.

$$\boxed{(f, g) = \int_a^b r(x) f(x) g(x)\, dx}$$

orthogonal system on $[a, b]$ w.r.t. $r(x) > 0$

$$\{ y_0(x), y_1(x), \cdots, y_n(x), \cdots \}$$

$$(y_n, y_m) = 0 \quad n \neq m$$

orthogonal series/expansion or generalized F-series

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x), \quad a_m = \frac{(f, y_m)}{\|y_m\|^2}$$

$$= a_0 y_0(x) + a_1 y_1(x) + \cdots$$

$$(f, y_0) = (a_0 y_0 + a_1 y_1 + \cdots, y_0) = a_0 (y_0, y_0) + a_1 \underbrace{(y_1, y_0)}_{=0} + a_2 \overbrace{(y_2, y_0)}^{=0} + \cdots$$

$$= a_0 \|y_0\|^2 \implies a_0 = \frac{(f, y_0)}{\|y_0\|^2}$$

$$(f, y_m) = \left(\sum_{m=0}^{\infty} a_m y_m, y_n \right) = \sum a_m (y_m, y_n) = a_n (y_n, y_n) = a_n \|y_n\|^2$$

Ex.1 Fourier-Legendre Series

$$P_2 = \frac{3}{2} x P_1 - \frac{1}{2} P_0 = \frac{3}{2}x^2 - \frac{1}{2}$$

$$P_3 = \frac{5}{3} x P_2 - \frac{2}{3} P_1 = \frac{5}{3} x \left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \frac{2}{3}x$$

Legendre Polynomials $\{P_0(x), P_1(x), \cdots, P_n(x), \cdots\}$

$$P_0 = 1,\quad P_1 = x,\quad P_2 = \frac{3}{2}x^2 - \frac{1}{2},\quad \cdots,\quad P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

3-term recursive relation

orthogonality $[-1, 1]$, $r(x) = 1$

$$(P_n, P_m) = \int_{-1}^{1} P_n(x) P_m(x)\,dx = \begin{cases} \dfrac{2}{2m+1} & n = m \\ 0 & n \neq m \end{cases}$$

(4) P505

$$\|P_m\|^2 = \frac{2}{2m+1}, \Longrightarrow \|P_m\| = \sqrt{\frac{2}{2m+1}}$$

Fourier-Legendre Series

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 P_0(x) + a_1 P_1(x) + \cdots$$

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x) P_m(x)\,dx \qquad a_m = \frac{(f, P_m)}{\|P_m\|^2} = \frac{\int_{-1}^{1} f(x) P_m(x)\,dx}{\frac{2}{2m+1}}$$

#2 on p509

$$f(x) = (x+1)^2 = \underline{x}^2 + \underline{2x}+\underline{1} = \underline{1 + 2x + x^2}$$

$$(x+1)^2 =$$

$$f(x) = (x+1)^2, \qquad a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)\,P_m(x)\,dx = ?$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,P_0(x)\,dx = \frac{1}{2}\int_{-1}^{1}(x+1)^2\,dx = \frac{1}{2}\cdot\frac{1}{3}(x+1)^3\Big|_{-1}^{1} = \frac{4}{3}$$

$$a_1 = \frac{3}{2}\int_{-1}^{1}(x+1)^2 x\,dx = \frac{3}{2}\int_{-1}^{1}\left[x + 2x^2 + x^3\right]dx$$

$$= \frac{3}{2}\cdot 2\cdot\frac{1}{3}x^3\Big|_{-1}^{1} = 1+1 = 2$$

$$a_2 = \frac{5}{2}\int_{-1}^{1}(1+2x+x^2)\left(\frac{3}{2}x^2 - \frac{1}{2}\right)dx = \frac{5}{2}\int_{-1}^{1}\left[\frac{3}{2}x^2 + \frac{3}{2}x^4 - \frac{1}{2} - \frac{1}{2}x^2\right]dx$$

$$= \cancel{\tfrac{5}{2}}\,5\int_{0}^{1}\left[x^2 + \frac{3}{2}x^4 - \frac{1}{2}\right]dx$$

$$= 5\left[\frac{1}{3} + \frac{3}{2}\cdot\frac{1}{5} - \frac{1}{2}\right] = 5\left[\frac{1}{3} + \frac{3}{10} - \frac{1}{2}\right] = \frac{\cancel{10+9-15}\;4}{\cancel{30}\;6}\cdot\cancel{5} = \frac{2}{3}$$

$$a_3 = \frac{7}{2} \int_{-1}^{1} \boxed{(x+1)^2} P_3(x)\, dx \qquad (x+1)^2 \stackrel{?}{=} a_0 P_0 + a_1 P_1 + a_2 P_2$$

$$= 0 \qquad 1 + 2x + x^2$$

$$\boxed{a_m = 0 \quad m \geq 3} \qquad 1 = P_0$$

$$x = P_0 \quad P_1$$

$$(x+1)^2 = \boxed{\frac{4}{3}1 + 2x + \frac{2}{3}\boxed{\left(\frac{3}{2}x^2 - \frac{1}{2}\right)}}$$

$$= 1 + 2x + x^2$$

Mean Square Convergence. Completeness.

$\{f_0(x), f_1(x), \cdots, f_k(x), \cdots\}$

- $\lim_{k\to\infty} f_k(x) = f(x) \iff \lim_{k\to\infty} \|f_k - f\| = 0$

- $\sum_{m=1}^{\infty} a_m y_m(x)$ converges to $f(x) \iff \lim_{k\to\infty} s_k(x) = f(x)$ where $s_k(x) = \sum_{m=1}^{k} a_m y_m(x)$

  $\Updownarrow$ $f(x) = \sum_{m=1}^{\infty} a_m y_m \iff \lim_{k\to\infty} \|s_k - f\| = 0$

Let S be a set of functions defined on $[a, b]$

and $\{y_0, y_1, \cdots, y_m, \cdots\}$ be an orthonormal set $\quad (f, g) = \int_a^b r f g \, dx$

- $\{y_0, y_1, \cdots\}$ is complete in $S \iff \forall f \in S, \ \exists \sum_{m=0}^{\infty} a_m y_m(x)$

  s.t. $f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$

Parseval Equality

$$\|f\|^2 = \sum_{m=0}^{\infty} a_m^2$$

assumption $\|y_m\| = 1$

Bessel's Inequality

$$\|f\|^2 \geq \sum_{m=0}^{k} a_m^2$$

$$\|f\|^2 = (f, f) = \left(\sum_{m=0}^{\infty} a_m y_m, \sum_{n=0}^{\infty} a_n y_n \right)$$

$$= \sum_{m=0}^{\infty} a_m \left(y_m, \sum_{n=0}^{\infty} a_n y_n \right) = \sum_{m=0}^{\infty} a_m \sum_n a_n (y_m, y_n)$$

$$= \sum_n a_n (y_n, y_n) = \sum_n a_n^2 \|y_n\|^2$$

$$\|f\|^2 = \sum_{m=0}^{\infty} a_m^2 \|y_m\|^2$$

$$\geq \sum_{m=0}^{k} a_m^2 \|y_m\|^2$$

Thrm (Completeness)

$$(f, y_m) = 0 \ \forall m \Longrightarrow \|f\| = 0 \xRightarrow{f \text{ is cont.}} f(x) \equiv 0$$

$$\left(\sum a_n y_n, y_m \right) = a_m (y_m, y_m)$$

#5 P509 §11.6

Prove that if <u>$f(x)$ is even</u>

$$\overset{?}{\Longrightarrow} f(x) = \sum_{m=0}^{\infty} a_m P_m(x) \overset{?}{=} \sum_{k=0}^{\infty} a_{2k} P_{2k}(x)$$

$a_{2k+1} \overset{?}{=} 0$ $f(x)$ is even

$a_{2k} = 0$ f is odd

§11.5, #5 on p503

$\underline{P_n}(\underline{\cos\theta})$ $n=0,1,\cdots$ $\boxed{\text{form}}$ an orthogonal set on $\underline{[0,\pi]}$

$\cos\theta = x$

$P_n(x)$ — on $[-1, 1]$