§11.7 Fourier Integral

periodic funct $\longrightarrow$ nonperiodic funct.

F-series $\longrightarrow$ F-integral

$f_L(x)$: $f_L(x+2L) = f_L(x)$

F-series $\left(\omega_n = \frac{\pi}{L}\right)$

$$f_L(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos \omega_n x + b_n \sin \omega_n x\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_L(v) \cos \omega_n v \, dv, \quad b_n = \frac{1}{L}\int_{-L}^{L} f_L(v) \sin \omega_n v \, dv.$$

F-integral

$$f(x) = \lim_{L \to \infty} f_L(x)$$

$$= \int_0^{\infty}\left[A(\omega)\cos \omega x + B(\omega)\sin \omega x\right] dx$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \cos \omega v \, dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \sin \omega v \, dv$$

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \omega_n x + b_n \sin \omega_n x \right)$$

$$= \frac{1}{2L} \int_{-L}^{L} f_L(v)\,dv + \frac{1}{L} \sum_{n=1}^{\infty} \left[\cos \omega_n x \underbrace{\boxed{\int_{-L}^{L} f_L(v) \cos \omega_n v\,dv}}_{A_L(\omega_n)\pi} + \sin \omega_n x \underbrace{\boxed{\int_{-L}^{L} f_L(v) \sin \omega_n v\,dv}}_{B_L(\omega_n)\pi} \right]$$

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L}$$

$$= \frac{1}{2L} \int_{-L}^{L} f_L(v)\,dv + \sum_{n=1}^{\infty} \left[\cos \omega_n x \, A_L(\omega_n) + \sin \omega_n x \, B_L(\omega_n) \right] \Delta\omega$$

$$\longrightarrow \int_0^{\infty} \left[A(\omega) \cos \omega x + B(\omega) \sin \omega x \right] d\omega$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v\,dv, \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin \omega v\,dv$$

Thrm (F-integral)

Assumptions (1) $f(x)$ is piecewise continuous in every finit interval

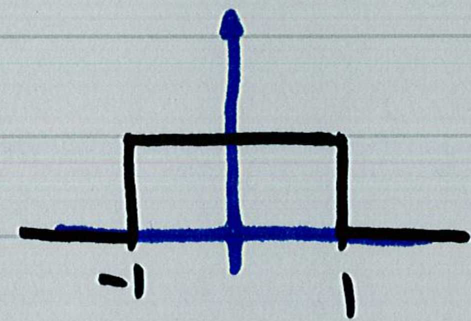
(2) at discont. pt x_0, $f'(x_0^-)$ and $f'(x_0^+)$ exist

(3) $\int_{-\infty}^{\infty} |f(x)|\, dx$ exist

$$\Longrightarrow \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega = \begin{cases} f(x) & \text{at cont. pt.} \\ \dfrac{f(x^-)+f(x^+)}{2} & \text{at discont. pt.} \end{cases}$$

Ex. 2 (Single Pulse. Sine Integral. Dirichlet's Discont. Factor. Gibbs Phenomenon.)

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$



$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w}\, dw$$

$$\int_0^\infty \frac{\cos wx \sin w}{w}\, dw = \begin{cases} \frac{\pi}{2} & x \in [0,1) \\ \frac{\pi}{4} & x = 1 \\ 0 & x > 1 \end{cases}$$

Dirichlet's Discont. factor

sine integral

$\underline{x=0}$

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin w}{w}\, dw = \lim_{u\to\infty} \int_0^u \frac{\sin w}{w}\, dw = \lim_{u\to\infty} Si(u)$$

$$Si(u) = \int_0^u \frac{\sin w}{w}\, dw$$

Fig. 282, 283

Fourier Cosine and Sine Integrals

- **F-cosine integral** $\quad f(x)$ is even $\Longrightarrow B(w)=0$

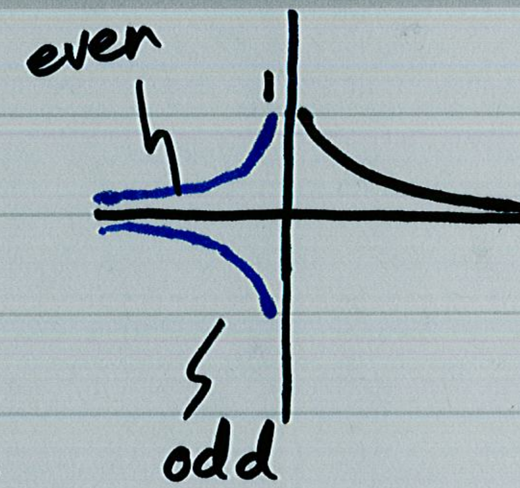
$$f(x)=\int_0^{\infty} A(w)\cos(wx)\,dw, \qquad A(w)=\frac{2}{\pi}\int_0^{\infty} f(v)\cos(wv)\,dv$$

- **F-sine integral** $\quad f(x)$ is odd $\Longrightarrow A(w)=0$

$$f(x)=\int_0^{\infty} B(w)\sin(wx)\,dw, \quad B(w)=\frac{2}{\pi}\int_0^{\infty} f(v)\sin(wv)\,dv$$

Ex. 3 Laplace Integrals

$$f(x) = e^{-kx} \quad \text{where } x > 0 \text{ and } k > 0$$



(a) $f \longrightarrow$ even function

$$\frac{\pi}{2k} e^{-kx} = \int_0^\infty \frac{\cos wx}{k^2 + w^2} \, dw$$

(b) $f \longrightarrow$ odd function

$$\frac{\pi}{2} e^{-kx} = \int_0^\infty \frac{w \sin wx}{k^2 + w^2} \, dw$$