$P_n(x)$ — L. poly.

$$\boxed{\int_{-1}^{1} P_n(x) P_m(x)\,dx = \begin{cases} * & n=m \\ 0 & n\neq m \end{cases}}$$

$$\underline{\left\{ P_n\left(\underline{\underline{\cos\theta}}\right) \right\}} \quad \theta \in [0,\pi]$$

$P_0 = 1$ $\qquad P_0(\cos\theta) = 1$

$x = \cos\theta$ $\qquad P_1 = x$ $\qquad P_1(\cos\theta) = \underline{\underline{\cos\theta}}$

$dx = -\sin\theta\, d\theta$ $\qquad 1 \to -1$

$$\int_0^{\pi} \underline{\sin\theta}\, P_n(\cos\theta) P_m(\cos\theta)\, \underline{d\theta} = 0 \quad \underline{n \neq m}$$

$$\|$$

$$\int_1^{-1} P_n(x) P_m(x) \left(-dx\right)$$

Chapter 6

$Y(s) = \mathcal{L}\{y\}$

$F(s)$

§12.12

Chapter 11.1 — 11.5

§11.7 Fourier Integral

periodic funct $\longrightarrow$ nonperiodic funct.

F-series $\longrightarrow$ F-integral

$f_L(x)$: $f_L(x+2L) = f_L(x)$

F-series $\left(\omega_n = \frac{n\pi}{L}\right)$

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \omega_n x + b_n \sin \omega_n x \right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_L(v) \cos \omega_n v \, dv, \quad b_n = \frac{1}{L}\int_{-L}^{L} f_L(v) \sin \omega_n v \, dv.$$

F-integral

$$f(x) = \lim_{L \to \infty} f_L(x)$$

$$= \int_0^{\infty} \left[A(\omega) \cos \omega x + B(\omega) \sin \omega x \right] d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \cos \omega v \, dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \sin \omega v \, dv$$

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \omega_n x + b_n \sin \omega_n x \right)$$

$$= \frac{1}{2L} \int_{-L}^{L} f_L(v)\,dv + \frac{1}{L} \sum_{n=1}^{\infty} \Big[\cos \omega_n x \underbrace{\boxed{\int_{-L}^{L} f_L(v) \cos \omega_n v\,dv}}_{A_L(\omega_n)\pi} + \sin \omega_n x \underbrace{\boxed{\int_{-L}^{L} f_L(v) \sin \omega_n v\,dv}}_{B_L(\omega_n)\pi} \Big]$$

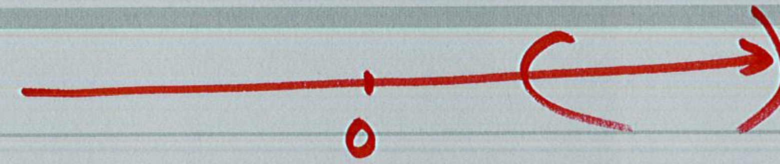
$$\Delta \omega = \omega_{n+1} - \omega_n = \frac{\pi}{L}$$

$$= \frac{1}{2L} \int_{-L}^{L} f_L(v)\,dv + \sum_{n=1}^{\infty} \left[\cos \omega_n x\, A_L(\omega_n) + \sin \omega_n x\, B_L(\omega_n) \right] \Delta \omega$$

$$\longrightarrow \int_0^{\infty} \left[A(\omega) \cos \omega x + B(\omega) \sin \omega x \right] d\omega$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v\,dv, \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin \omega v\,dv$$

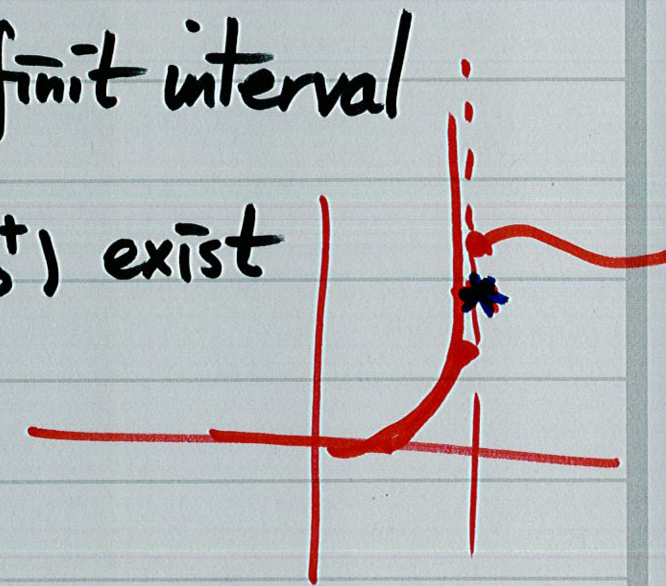
Thrm (F-integral)



Assumptions

(1) $f(x)$ is piecewise continuous in every finit interval

(2) at discont. pt x_0, $f'(x_0^-)$ and $f'(x_0^+)$ exist

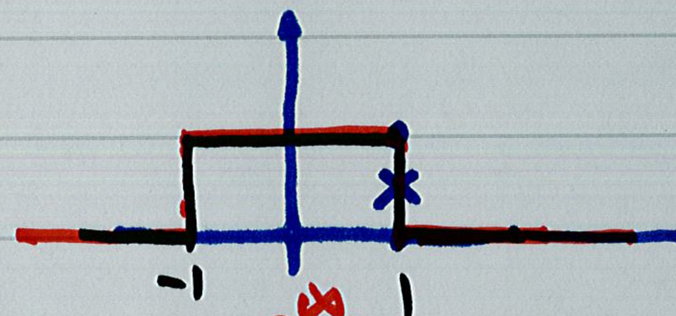


(3) $\int_{-\infty}^{\infty} |f(x)|\,dx$ exist

$$\Longrightarrow \int_0^{\infty} \left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right] d\omega = \begin{cases} f(x) & \text{at cont. pt.} \\ \dfrac{f(x^-)+f(x^+)}{2} & \text{at discont. pt.} \end{cases}$$

Ex.2 (Single Pulse. Sine Integral. Dirichlet's Discont. Factor. Gibbs Phenomenon.)

$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$



$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos wx\,dx$$

$$= \frac{1}{\pi}\int_{-1}^{1} \cos wx\,dx = \left.\frac{\sin wx}{\pi w}\right|_{-1}^{1}$$

$$f(x) = \int_0^{\infty}\left[A(w)\cos wx + \cancel{B(w)\sin wx}\right]dw$$

$$\frac{1}{\pi w}\left[\sin w - \sin(-w)\right] = \frac{2\sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi}\int_{-1}^{1}\sin wx\,dx = -\left.\frac{\cos wx}{\pi w}\right|_{-1}^{1} = 0$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\cos wx \sin w}{w}\,dw$$

$$\int_0^{\infty}\frac{\cos wx \sin w}{w}\,dw = \begin{cases} \frac{\pi}{2} & x \in [0,1) \\ \frac{\pi}{4} & x = 1 \\ 0 & x > 1 \end{cases}$$

Dirichlet's Discont. factor

$x=0$

$$\frac{\pi}{2} = \int_0^{\infty}\frac{\sin w}{w}\,dw = \lim_{u\to\infty}\int_0^{u}\frac{\sin w}{w}\,dw = \lim_{u\to\infty} Si(u)$$

sine integral

$$Si(u) = \int_0^{u}\frac{\sin w}{w}\,dw$$

Fig. 282, 283

Fourier Cosine and Sine Integrals

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin wx\,dx$$

- **F-cosine integral** $\quad f(x)$ is even $\Longrightarrow B(w) = 0$

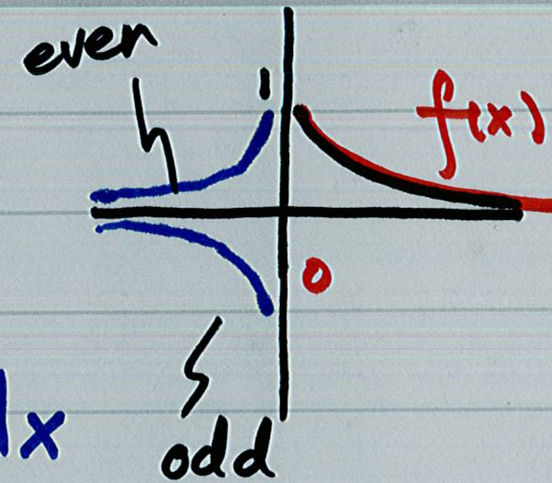
$$f(x) = \int_0^\infty A(w)\cos(wx)\,dw, \qquad A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos(wv)\,dv$$

- **F-sine integral** $\quad f(x)$ is odd $\Longrightarrow A(w) = 0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos wx\,dx$

$$f(x) = \int_0^\infty B(w)\sin(wx)\,dw, \qquad B(w) = \frac{2}{\pi}\int_0^\infty f(v)\sin(wv)\,dv$$

Ex. 3 Laplace Integrals

$$f(x) = e^{-kx} \quad \text{where } x>0 \text{ and } k>0$$



$+\infty$

$$A(w) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos wx\,dx = \frac{2}{\pi}\int_0^{\infty} e^{-kx}\cos wx\,dx$$

(a) $f \longrightarrow$ even function

$$\frac{\pi}{2k}e^{-kx} = \int_0^{\infty}\frac{\cos wx}{k^2+w^2}\,dw$$

(b) $f \longrightarrow$ odd function

$$\frac{\pi}{2}e^{-kx} = \int_0^{\infty}\frac{w\sin wx}{k^2+w^2}\,dw$$