Fourier Integral

$f(x)$ on $(-\infty, +\infty)$ not periodic

$$\boxed{f(x) = \int_0^{\infty} \left[A(w) \cos wx + B(w) \sin wx \right] dx}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right] \quad \text{on } (-\pi, \pi)$$

$$\boxed{\begin{aligned} A(w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos wx \, dx \\ B(w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin wx \, dx \end{aligned}}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

- **f is even**

$$f(x) = \int_0^\infty A(w)\cos wx\,dx, \quad A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos wx\,dx$$

- f is odd

$$f(x) = \int_0^\infty B(w)\sin wx\,dx, \quad B(w) = \frac{2}{\pi}\int_0^\infty f(x)\sin wx\,dx$$

§11.8 Fourier Cosine and Sine Transforms

<u>Laplace Transform</u> $\quad F(s) = \int_0^\infty e^{-st} f(t)\,dt$

• <u>F - Cosine Transform</u> $\quad \underline{A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos(wv)\,dv}$

$$\underline{f(x) = \int_0^\infty A(w)\cos(wx)\,dw} = \sqrt{\frac{2}{\pi}}\int_0^\infty \boxed{\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\cos(wy)\,dy\right]}\cos(wx)\,dw$$

$\hat{f}_c(w)$

$$\mathcal{F}_c(f) = \hat{f}_{\underline{c}}(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(wx)\,dx$$

$$f(x) = \mathcal{F}_c^{-1}(\hat{f}_c) = \sqrt{\frac{2}{\pi}}\int_0^\infty \boxed{\hat{f}_c(w)}\cos(wx)\,dx$$

$$f(x) = \mathcal{F}_c^{-1}(\hat{f}_c) = \underline{\mathcal{F}_c^{-1}}\left(\underline{\mathcal{F}_c}(f)\right)$$

• $\mathcal{F}$-sine transform

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(wx)\,dx = \mathcal{F}_s(f)$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin(wx)\,dw = \mathcal{F}_s^{-1}(\hat{f}_s)$$

Ex. 1 $f(x) = \begin{cases} k & x \in (0, a) \\ 0 & x > a \end{cases}$ $\mathcal{F}_c(f) = ?$, $\mathcal{F}_s(f) = ?$

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx\,dx = \sqrt{\frac{2}{\pi}} k \int_0^a \cos wx\,dx = \sqrt{\frac{2}{\pi}} k \left. \frac{\sin wx}{w} \right|_0^a$$

$$= \sqrt{\frac{2}{\pi}} k \frac{\sin aw}{w} = \hat{f}_c(w)$$

$$\mathcal{F}_s(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx\,dx = \sqrt{\frac{2}{\pi}} k (-1) \left. \frac{\cos wx}{w} \right|_0^a$$

Ex. 2

$$\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty \underline{e^{-x}\cos wx\,dx}$$

$$\boxed{\begin{array}{ll} \underline{u} = e^{-x} & \underline{v'} = \cos wx \\ u' = -e^{-x} & v = \frac{\sin wx}{w} \end{array}}$$

$$= \sqrt{\frac{2}{\pi}} \left[e^{-x}\frac{\sin wx}{w} \Big|_0^\infty + \frac{1}{w}\int_0^\infty \underline{e^{-x}\sin wx\,dx} \right]$$

$u = e^{-x}$ $\quad v' = \sin wx$

$u' = -e^{-x}$ $\quad v = -\frac{\cos wx}{w}$

$$\cancel{e^{-\infty} \sin w\infty} \le e^{-\infty} = 0$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{w} \underline{\left[-e^{-x}\frac{\cos wx}{w} \Big|_0^\infty - \frac{1}{w}\int_0^\infty e^{-x}\cos wx\,dx \right]} \right]$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{w^2}\left[\overset{1}{\cancel{\#}} - \cancel{\#}\int_0^\infty \underline{e^{-x}\cos wx\,dx} \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w^2} - \frac{1}{w^2}\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x}\cos wx\,dx$$

$$\cancel{\mathcal{F}_c(e^{-x}) = \frac{1}{\sqrt{2\pi}} \frac{1}{w^2}}$$

$$\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w^2} - \frac{1}{w^2}\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w^2} - \frac{1}{w^2} \cdot \mathcal{F}_c(e^{-x})$$

$$\left(1 + \frac{1}{w^2}\right) \mathcal{F}_c(e^{-x}) = \frac{1}{w^2}\sqrt{\frac{2}{\pi}}$$

$$\parallel$$

$$\frac{1+w^2}{w^2} \qquad \mathcal{F}_c(e^{-x}) = \frac{\sqrt{\frac{2}{\pi}}}{1+w^2}$$

Linearity, Transforms of Derivatives

Assumptions (1) $f(x)$ is absolutely integrable $\iff \int_0^\infty |f(x)|\,dx$ exists

(2) $f(x)$ is p. cont. on every finite interval

• **Linearity** $\quad \mathcal{F}_{\substack{c\\s}}\left(a f + b g\right) = a\,\mathcal{F}_{\substack{c\\s}}(f) + b\,\mathcal{F}_{\substack{c\\s}}(g)$

(3) f' is p. cont. on every finite interval

(4) $\lim_{x\to\infty} f(x) = 0$ $\qquad \mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

• **Derivatives** $\quad \mathcal{F}_c\left(f'(x)\right) = w\,\mathcal{F}_s\left(f(x)\right) - \sqrt{\frac{2}{\pi}}\,f(0)$

$\mathcal{F}_s\left(f'(x)\right) = -w\,\mathcal{F}_c\left(f(x)\right)$

$$\mathcal{F}_c(f'(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x)\cos wx\,dx$$

$u = \cos wx, \quad v' = f'$

$u' = -w\sin wx, \quad v = f$

$$= \sqrt{\frac{2}{\pi}} \left[f(x)\cos wx \Big|_0^\infty + w\int_0^\infty f(x)\sin wx\,dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\cancel{f(\infty)}\cos w\infty - f(0) \right] + w\,\underline{\mathcal{F}_s(f)}$$

0

$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx\,dx$

$$\boxed{\lim_{x\to\infty} f(x) = 0}$$

$$\mathcal{F}_c(f'') = -w^2\mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}}\, f'(0)$$

$$\mathcal{F}_c(f'') = \mathcal{F}_c\left((f')'\right)$$

$$= w\mathcal{F}_s(f') - \sqrt{\frac{2}{\pi}}\, f'(0)$$

$$= -w^2\mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}}\, f'(0)$$

$$\mathcal{F}_s(f'') = -w^2\mathcal{F}_s(f) + \sqrt{\frac{2}{\pi}}\, w f(0)$$

Ex. 3 $\mathcal{F}_c(e^{-ax}) = ?$ where $a > 0$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos wx\,dx$$

$$\mathcal{F}_c(f') = w\mathcal{F}_s(f) - \sqrt{\frac{2}{\pi}}$$

$$\mathcal{F}_c(f') = -a\mathcal{F}_c(f)$$

$f = e^{-ax}$ $\quad f' = -ae^{-ax} = -af$

$$\boxed{f'' = a^2 f}$$

$$\mathcal{F}_c(f'') = -w^2\mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}}\,(-a)$$

$$\mathcal{F}_c(f'') = a^2\mathcal{F}_c(f)$$

$$= -w^2\mathcal{F}_c(f) + a\sqrt{\frac{2}{\pi}}$$

$$\mathcal{F}_c(f) = \frac{a\sqrt{2/\pi}}{w^2 + a^2}$$

$$\mathcal{F}_c\left(e^{-x}\right) = \frac{\sqrt{2/\pi}}{w^2+1} = \hat{f}_c(w)$$

$f(x) = e^{-x}$

$$\mathcal{F}_c^{-1}\left(\hat{f}_c(w)\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos wx \, dw$$

$u = \cos wx \quad v' = \frac{1}{w^2+1}$

$u' = \sin wx \quad v =$

$$\boxed{= \frac{2}{\pi}\int_0^\infty \frac{\cos wx}{w^2+1}\,dw}$$

$$= e^{-x}$$