§11.8 #2

$\underline{\underline{\hat{f}_c(w)}}$ $\underline{\#1}$ $\frac{\cos w}{w}$, $\frac{\cos 2w}{w}$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \underline{\hat{f}_c(w)} \cos wx \, dw$$

$$\hat{f}_c(w) = \cancel{*}\frac{2\sin w - \sin 2w}{w}$$

$$= * \int_0^\infty \frac{\sin w \cos wx}{w} dw$$

$\underline{\text{p514}}$ (7)

$$\int_0^\infty \frac{\cos wx \sin w}{w} dw = \begin{cases} \frac{\pi}{2} & \underline{[0,1)} \\ \frac{\pi}{4} & \underline{x=1} \\ 0 & \underline{x>1} \end{cases}$$

$u = 2w$

$$\int_0^\infty \frac{\sin \boxed{2w} \cos wx}{w} dw \quad u\left(\frac{x}{2}\right)$$

D.D.F.

§11.9 Fourier Transform. Discrete and Fast Fourier Transform.

Complex Form of the Fourier Integral

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\, e^{iw(x-v)}\,dv\,dw$$

$$f(x) = \int_{0}^{\infty}\left[A(w)\cos wx + B(w)\sin wx\right]dw$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\left[\underbrace{\cos wv\cos wx + \sin wv\sin wx}_{\cos(wx-wv)}\right]dv\right]dw$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(wx-wv)\,dv\right]dw$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\Big[\underbrace{\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\,dv}\Big]\,dw \qquad \cos w(x-v)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} f(v)\cos(wx - wv)\,dv\Big]\,dw$$

$$+\, i\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} f(v)\sin(wx - wv)\,dv\Big]\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\, e^{i(wx - wv)}\,dv\,dw$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\, e^{iw(x-v)}\, dv\, dw.$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\, e^{-iwv}\, dv\right]}\, e^{iwx}\, dw$$

F-transform

$$\mathcal{F}(f) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$\mathcal{F}^{-1}(\hat{f}(w)) = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw$$

inverse F-transform

F-cosine transform

$$\hat{f}_{\underset{s}{\underline{c}}}(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \underset{\sin wx}{\underline{\cos wx}} \, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_{\underset{s}{\underline{c}}}(w) \underset{\sin wx}{\underline{\cos wx}} \, dw$$

F-sine transform

Fourier Transform $\quad \hat{f}(w) = \mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$

$$= \boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) [\boxed{\cos wx}} - i \sin wx] dx$$

Inverse F-transform $\quad f(x) = \mathcal{F}^{-1}(\hat{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dx$

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos wx \, dx$$

f(x) – even $\quad$ F-tranform = F-cosine t.

f(x) – odd $\quad$ F-tranforn = F-sine t.

<u>Ex. 1</u> $f(x) = \begin{cases} 1 & |x|<1 \\ 0 & \text{otherwise} \end{cases}$ $\quad \hat{f}(w) = ?$

$e^{iw} = \cos w + i \sin w$

$e^{-iw} = \cos w - i \sin w$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, \cancel{\cos wx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx}\, \cancel{\cos wx}\, dx$$

$$\cancel{= \frac{1}{\sqrt{2\pi}} \frac{\sin wx}{w} \Big|_{-1}^{1} = \frac{2 \sin w}{\sqrt{2\pi}\, w} = \frac{\sqrt{2} \sin w}{\sqrt{\pi}\, w}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{i}{-i^2 w} e^{-iwx} \Big|_{-1}^{1} = i \frac{1}{\sqrt{2\pi}\, w} \Big[\underbrace{e^{-iw} - e^{iw}}_{-2i \sin w} \Big]$$

$$= \frac{\sqrt{2} \sin w}{\sqrt{\pi}\, w}$$

Ex. 2 $f(x) = \begin{cases} e^{-ax} & x>0 \\ 0 & x<0 \end{cases}$ $\quad (a>0)$

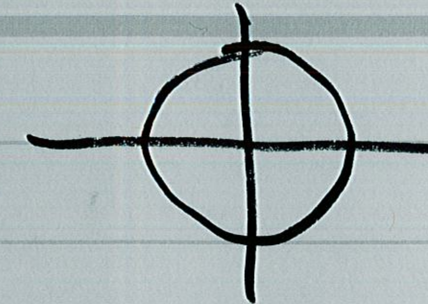
$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(a+iw)x}\, dx$$

$a^2-(iw)^2$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{-1}{a+iw}\, e^{-(a+iw)x}\Big|_{0}^{\infty}$$

$$= \frac{(a-iw)}{\sqrt{2\pi}(a+iw)(a-iw)} = \frac{a-iw}{\sqrt{2\pi}\left(a^2+w^2\right)}$$



$$\lim_{x\to\infty} e^{-(a+iw)x}$$

$$= \lim e^{-ax}\cdot e^{-iwx}$$

$$= 0$$

Ex.2 $f(x)=\begin{cases} e^{-ax}, & x>0 \\ 0, & x<0 \end{cases}$ $(a>0)$ $\mathcal{F}(f)=?$

$u=e^{-i\omega x}$ $v'=f'$

$u'=-i\omega e^{-i\omega x}$ $v=f$

$$\mathcal{F}(f')=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-i\omega x}\,dx$$

$$=\boxed{f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty}}+i\omega\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

?

$$=i\omega\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}\,dx$$

Properties

$$\mathcal{F}(af+bg)=a\mathcal{F}(f)+b\mathcal{F}(g)$$

$f(\infty)=0$

$f(-\infty)=0$

$$\boxed{\mathcal{F}(f'(x))=i w\mathcal{F}(f)}$$

$$\lim_{x\to\pm\infty} f(x)=0$$

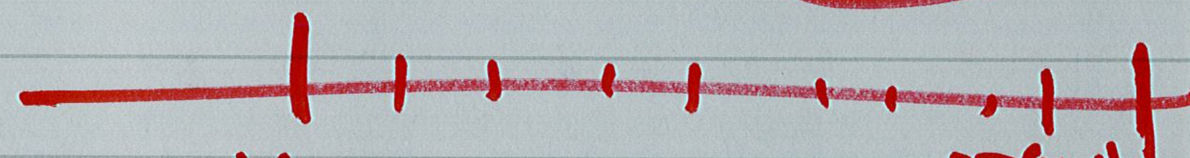
$$\mathcal{F}(f*g)=\sqrt{2\pi}\,\mathcal{F}(f)\,\mathcal{F}(g)$$

$$f*g=\int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy$$

$$\frac{1}{\sqrt{2\pi}}f*g=\mathcal{F}^{-1}\{\hat{f}\,\hat{g}\}$$

$f(x)$ $\quad x_k = \frac{2\pi k}{N}$ $\quad \underbrace{k = 0, 1, \cdots, N-1}_{N}$

x_0



$x_0 = 0$ $\quad \frac{2\pi(N-1)}{N}$ $\quad 2\pi$

$O(N^2)$

FFT

$O(N)\log_2 N$