Chapter 12 Partial Differential Equations

§12.1 Basic Concepts of PDEs

order, linear
nonlinear

Examples

$u(x,t)$

wave eq. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$ hyperbolic PDE

$u(x,y,z,t)$

2^{nd}-order

heat eq. $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ parabolic PDE

$u(x,y)$

Laplace eq. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ elliptic PDE

$\Delta u = 0$

$$F\left(\underline{u}, \underline{\frac{\partial u(x_1, \cdots, x_n)}{\partial x_i}}, \underline{\frac{\partial^2 u}{\partial x_i \partial x_j}}, \cdots, \underline{\frac{\partial^n u}{\partial x_i \cdots \partial x_j}}\right) = 0$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \underline{\underline{u^2}}$$

$c(u)$

$+ \sin u$

$$(x_1, \cdots, x_4) = (x, y, z, t)$$

$$cu + e^{\frac{\partial u}{\partial x_i}}$$

homog.

nonhomog.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \underline{\underline{f(x,y)}}$$

$$u''(x) + 3u'(x) + u(x) = 0$$

$f(x)$

Navier-Stokes Eq

$$\begin{cases} -\Delta \vec{u} + \nabla p = f(x,y,z) & \text{in } \Omega \\ \operatorname{div} \vec{u} = 0 \end{cases}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\vec{u} = (u_1, u_2, u_3)$$

$$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$$

$$\operatorname{div} \vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$$

Superposition

u_1 and u_2 are solutions of a homog. linear PDEs $\Longrightarrow$ $c_1 u_1 + c_2 u_2$ — solution

Ex. 1 $\quad \dfrac{\partial^2 u(x,y)}{\partial x^2} - u(x,y) = 0 \Longrightarrow u(x,y) = A(y)e^{x} + B(y)e^{-x}$

$u''(x) - u(x) = 0$

$s^2 - 1 = 0 \Rightarrow s = \pm 1 \qquad u(x) = c_1 e^{-x} + c_2 e^{x}$

$$\boxed{u(x,y) = c_1(y)e^{-x} + c_2 y e^{x}}$$

Ex. 2 $$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial u}{\partial x}$$

$$p = \cancel{\frac{\partial u}{\partial y}}\frac{\partial u}{\partial x} \qquad \frac{\partial p}{\partial y} = -p \qquad \frac{dp}{p} = -dy$$

$$\ln p = -y + c(x)$$

$$\frac{\partial u}{\partial x} = p(x,y) = e^{-y} \cdot e^{c(x)} = C(x)\, e^{-y}$$

$$u(x,y) = \int C(x)\, e^{-y}\, dx + D(y)$$

$$= f(x)\, e^{-y} + D(y)$$

#3 $u = \cos 4t \sin 2x$ — $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

$$\frac{\partial u}{\partial t} = -4 \sin 4t \sin 2x$$

$$\boxed{u_{tt} = -16 \cos 4t \sin 2x} = 4 u_{xx} \quad \Rightarrow c = 2$$

$$u_x = 2 \cos 4t \cos 2x$$

$$u_{xx} = -4 \cos 4t \sin 2x$$

$$-\cos 4t \sin 2x = \frac{1}{4} u_{xx}$$

#6 $u = e^{-t}\sin x$

$$\boxed{\frac{\partial u}{\partial t} = -e^{-t}\sin x} = \frac{\partial^2 u}{\partial x^2}$$

$$\cancel{\frac{\partial^2 u}{\partial t^2} = e^{-t}\sin x} = \cancel{-\frac{\partial^2 u}{\partial x^2}}$$

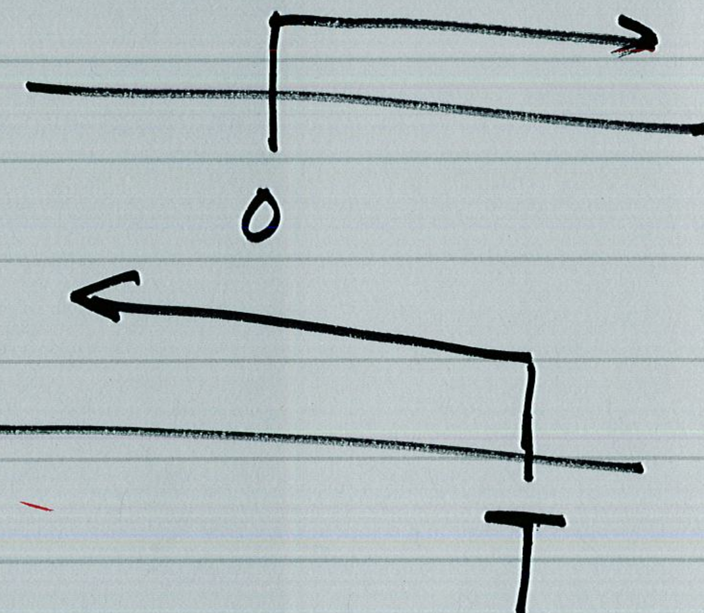
$$\frac{\partial u}{\partial x} = e^{-t}\cos x$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = -e^{-t}\sin x} \Rightarrow$$

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

$c=1$

$t \geq 0$



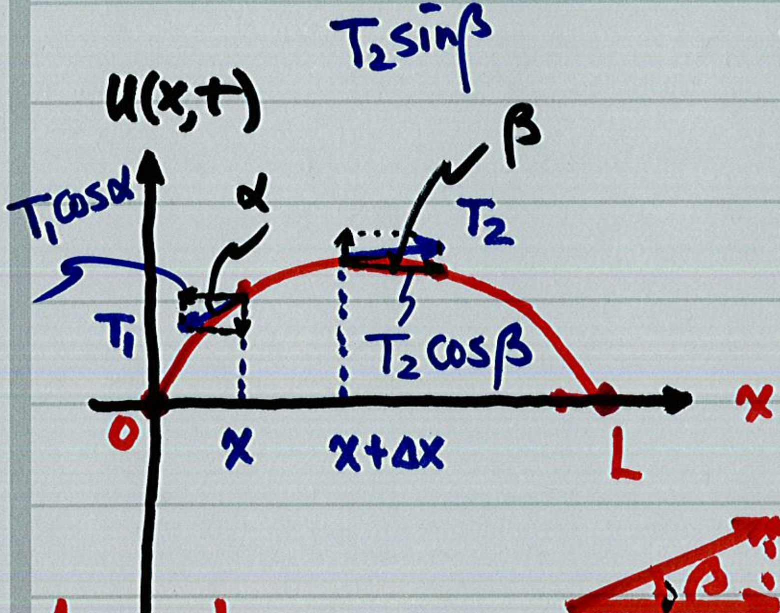
#19 $\underline{u_y + y^2 u = 0}$ $u(x, y)$

$$u' + y^2 u = 0$$

$$\frac{du}{dy} = -y^2 u \qquad \frac{du}{u} = -y^2 dy$$

§12.2 Modeling: Vibrating String, Wave Eq.

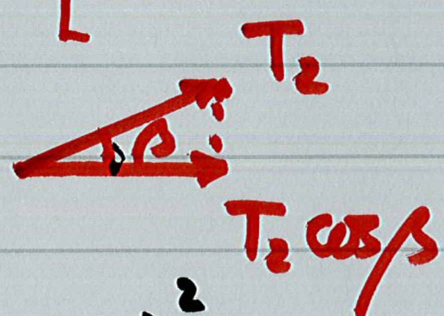
derivation of $$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} \Longrightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$



$$T_1\cos\alpha = T_2\cos\beta = T$$

$$T_2\sin\beta - T_1\sin\alpha = ma = \rho\Delta x\frac{\partial^2 u}{\partial t^2}$$

$$T$$



$$\frac{\tan\beta - \tan\alpha}{\Delta x} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\|$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) \leftarrow \frac{1}{\Delta x}\left[\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)\right]$$

Assumptions

- homog. string
- small displacement
- neglect gravitational force

§12.3 Solution by Separating Variables.

$$\begin{cases} u_{tt} = c^2 u_{xx} \qquad c^2 = \dfrac{T}{\rho} & \text{wave eq.} \\ u(0,t)=0 , \quad u(L,t)=0 & \text{BCs} \\ u(x,0)=f(x) , \quad u_t(x,0)=g(x) & \text{ICs} \end{cases}$$

initial deflection   initial velocity

Step 1 (method of ~~sp~~separating variables)

$$u(x,t)=F(x)G(t)$$

Step 2 (BCs)

$$\begin{cases} 0 = u(0,t) = F(0)\,G(t) \\ 0 = u(L,t) = F(L)\,G(t) \end{cases}$$