§12.10 Laplace in Polar Coordinates. Circular Membrane. Fourier-Bessel Series.

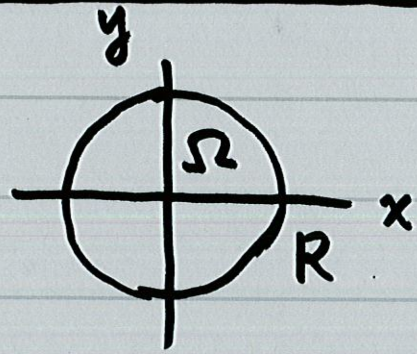
Laplace in Polar Coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = y/x \end{cases}$$

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} =$$

Circular Membrane

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), \quad c^2 = T/\rho$$



$u(r,t)$ — solution that is radially sym $\left(u_{\theta\theta} = 0\right)$

a membrane of radius R

$$\begin{cases} u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r\right) & \text{in } \Omega \times [0, +\infty) \\ u(R,t) = 0 & \forall t \geq 0 \\ u(r,0) = f(r), \quad u_t(r,0) = g(r) \end{cases}$$

Step 1

$$u(r,t) = W(r)\, G(t)$$

Bessel's eq. with $\nu = 0$ $$\frac{d^2W}{ds^2} + \frac{1}{s}\frac{dW}{ds} + W = 0$$

Bessel's functions $J_0(s)$ and $Y_0(s)$ — solutions of 1^{st} and 2^{nd} kind.

- since $Y_0(0) = \infty$ and $u(r,t)$ must be finite $\Longrightarrow$ $Y_0(s)$ cannot be used.

Step 2 (BCs) $W(r) = J_0(s) = J_0(kr)$

$$0 = u(R,t) = W(R)\,G(t) \Longrightarrow 0 = W(R) = J_0(kR)$$

- zeros of J_0: $J_0(\alpha_m) = 0$ for $m = 1, 2, \cdots$

where $\alpha_1 = 2.4048,\ \alpha_2 = 5.5201,\ \cdots$

$$0 = J_0(kR) \implies kR = \alpha_m \implies k_m = \frac{\alpha_m}{R}, \quad m = 1, 2, \cdots$$

$$0 = J_0(\alpha_m)$$

$$W_m(r) = J_0(k_m r) = J_0\left(\frac{\alpha_m}{R} r\right)$$

- eigenvalues $\qquad \lambda_m = c k_m = \dfrac{c\alpha_m}{R}$

- eigenfunctions $\qquad G_m(t) = A_m \cos \lambda_m t + B_m \sin \lambda_m t$

$$u_m(r,t) = \left(A_m \cos \lambda_m t + B_m \sin \lambda_m t\right) J_0(k_m r)$$

m^{th} normal mode

frequency: $\dfrac{\lambda_m}{2\pi}$ cycles per unit time

Fig. 308, 309

step 3

$$u(r,t) = \sum_{m=1}^{\infty} \left(A_m \cos \lambda_m t + B_m \sin \lambda_m t \right) J_0 \left(\frac{\alpha_m}{R} r \right)$$

$$f(r) = u(r,0)$$