§12.10 Laplace in Polar Coordinates. Circular Membrane.

Fourier-Bessel Series.

$u(x,y) = u(r,\theta)$

$\frac{\partial x}{\partial r} = x_r = \cos\theta, \quad x_\theta = -r\sin\theta$

$y_r = \sin\theta, \quad y_\theta = r\cos\theta$

Laplace in Polar Coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = y/x \end{cases}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(\ )^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta \qquad \frac{\partial r}{\partial y} = \sin\theta$$

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

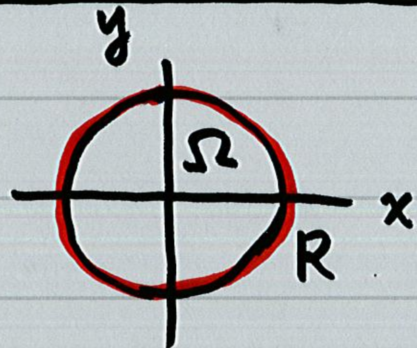
$$\frac{\partial \theta}{\partial x} =$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$\frac{\partial \theta}{\partial y}$$

Circular Membrane

$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right)$, $c^2 = T/\rho$



$u(r,t)$ — solution that is radially sym $\left(u_{\theta\theta}=0\right)$

a membrane of radius R

$$\begin{cases} u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r\right) & \text{in } \Omega \times [0,+\infty) \\ u(R,t) = 0 & \forall t \geq 0 \\ u(r,0) = f(r), \quad u_t(r,0) = g(r) & \end{cases}$$

Step 1 $\quad u(r,t) = W(r)\,G(t)$

$$u_{tt} = W\ddot{G} = c^2\left(W''G + \frac{1}{r}W'G\right) = c^2\left(W'' + \frac{1}{r}W'\right)G(t) \quad \Big/ c^2 G W$$

$$\frac{\ddot{G}(t)}{c^2 G(t)} = \frac{W''(r) + \frac{1}{r}W'(r)}{W(r)} = -k^2$$

$$0 = W'' + \frac{1}{r} W' + k^2 W \xrightarrow{r = ks} W(r) = W(ks)$$

$$\boxed{W''(s) + \frac{1}{s} W'(s) + W(s) = 0}$$

Bessel's Eq. $\nu = 0$

$J_0(s)$ and $Y_0(s)$

$$W(s) = c_1 J_0(s) + c_2 \cancel{Y_0(s)}$$

$$Y_0(0) = \infty$$

$$0 = \ddot{G} + (ck)^2 G$$

$$= \ddot{G} + \lambda^2 G$$

$$s^2 + \lambda^2 = 0 \qquad s = \pm \lambda i$$

$$G(t) = A_m \cos \lambda_m t + B_m \sin \lambda_m t$$

Bessel's eq. with $\nu = 0$

$$\boxed{\frac{d^2W}{ds^2} + \frac{1}{s}\frac{dW}{ds} + W = 0}$$

Bessel's functions $J_0(s)$ and $Y_0(s)$ — solutions of 1^{st} and 2^{nd} kind.

- since $Y_0(0) = \infty$ and $u(r,t)$ must be finite $\Longrightarrow$ $Y_0(s)$ cannot be used.

$\sin(ka) = 0$

$ka = n\pi,\ n = 1, 2, \cdots$

Step 2 (BCs) $\quad W(r) = J_0(s) = J_0(kr)$

$$0 = u(R,t) = W(R)\,G(t) \Longrightarrow 0 = W(R) = \underline{J_0(kR)}$$

$kR = \alpha_m$

$k_m = \dfrac{\alpha_m}{R}$

- zeros of J_0 : $\quad J_0(\alpha_m) = 0$ for $m = 1, 2, \cdots$

where $\alpha_1 = 2.4048,\ \alpha_2 = 5.5201, \cdots$

$$0 = J_0(kR) \qquad \Longrightarrow \qquad kR = \alpha_m \qquad \Longrightarrow \qquad k_m = \frac{\alpha_m}{R}, \quad m = 1, 2, \cdots$$

$$0 = J_0(\alpha_m)$$

$$W_m(r) = J_0(k_m r) = J_0\left(\frac{\alpha_m}{R} r\right)$$

- eigenvalues $\qquad \lambda_m = c k_m = \dfrac{c\alpha_m}{R}$

- eigenfunctions $\qquad G_m(t) = A_m \cos \lambda_m t + B_m \sin \lambda_m t$

$$u_m(r,t) = \left(A_m \cos \lambda_m t + B_m \sin \lambda_m t\right) J_0(k_m r)$$

m^{th} normal mode

Fig. 308, 309 $\qquad$ frequency: $\dfrac{\lambda_m}{2\pi}$ cycles per unit time

Step 3

$$u(r,t) = \sum_{m=1}^{\infty} \left(\underline{A_m} \cos \lambda_m t + \underline{B_m} \sin \lambda_m t \right) J_0\left(\frac{\alpha_m}{R} r\right)$$

$$f(r) = u(r,0) = \sum_{m=1}^{\infty} A_m \underline{J_0}\left(\frac{\alpha_m}{R} r\right)$$

$$\xRightarrow{\text{P507}} \quad A_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R r f(r) J_0\left(\frac{\alpha_m}{R} r\right) dr$$

$$g(r) = u_t(r,0)$$

Ex. 1 $R = 1$ ft, $\rho = 2$ slugs/ft^2, $T = 8$ lb/ft

$f(r) = 1 - r^2$ ft, $g(r) = 0$

Solution $c^2 = \frac{T}{\rho} = 4$ ft^2/sec^2

$$A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r(1-r^2)\, J_0(\alpha_m r)\, dr \qquad \text{(sec. 11.6)}$$

$$= \frac{4\, J_2(\alpha_m)}{\alpha_m^2\, J_1^2(\alpha_m)}$$

$$= \frac{8}{\alpha_m^3\, J_1(\alpha_m)} \qquad \left[J_2(\alpha_m) = \frac{2}{\alpha_m} J_1(\alpha_m) - J_0(\alpha_m) = \frac{2}{\alpha_m} J_1(\alpha_m) \right]$$