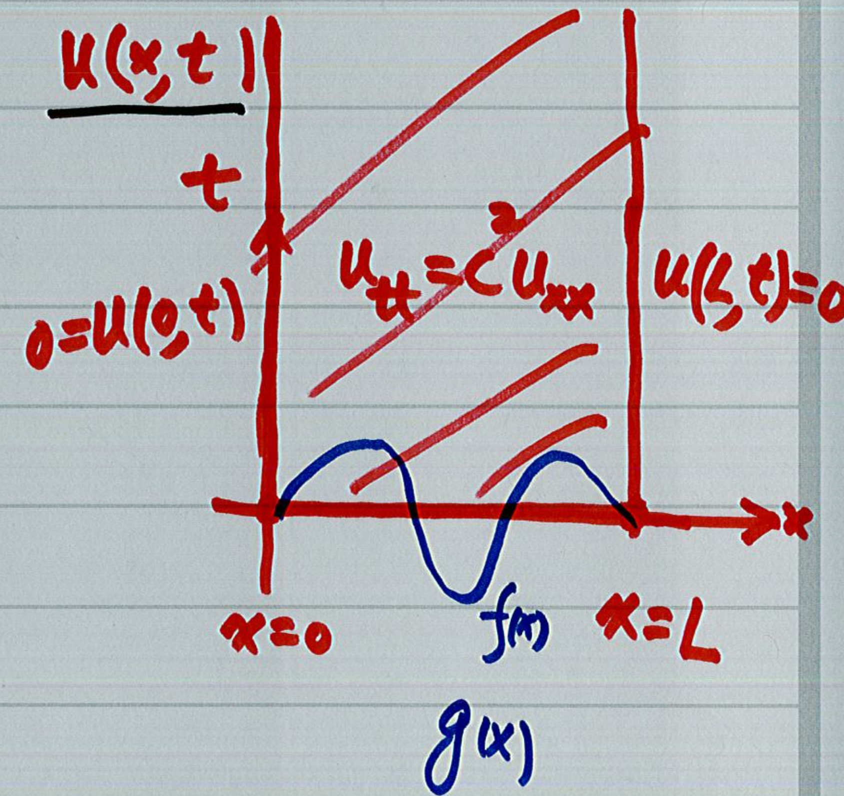
§12.3 Solution by Separating Variables.

step 1 $u_{tt} = c^2 u_{xx}$ $\quad c^2 = \frac{T}{\rho}$ wave eq.

step 2 $u(0,t)=0, \quad u(L,t)=0$ BCs

step 3 $u(x,0)=f(x), \quad u_t(x,0)=g(x)$ ICs

initial deflection, initial velocity



Step 1 (method of separating variables)

$$u(x,t) = F(x)G(t)$$

$u_t = F\dot{G}, \quad u_{tt} = F\ddot{G}$ $\qquad \dot{G}(t) = \frac{dG}{dt}$

$u_x = F'G, \quad u_{xx} = F''G$ $\qquad F'(x) = \frac{dF}{dx}$

$$F\ddot{G} = c^2 F''G \quad \forall (x,t) \in [0,L] \times [0,+\infty)$$

$$\frac{F(x)\,\ddot{G}(t) = c^2 F''(x)\,G(t)}{c^2 F(x)\,G(t)} \qquad \forall\, x, t$$

$F(x)$

$$+k = \frac{\ddot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} \equiv \not{\text{const}} = +k$$

$G(t)$

$$F''(x) - k F(x) = 0, \qquad x \in [0, L]$$

$$\ddot{G}(t) - c^2 k\, G(t) = 0, \qquad t \in [0, +\infty)$$

$u_{tt} = a(x,t)\, u_{xx}$

$a(x,t) \neq c(x)\, d(t)$

$$\frac{F(x)\,\ddot{G}(t) = a(x,t)\, F''(x)\, G(t)}{F(x)\, G(t)}$$

$$\frac{\ddot{G}(t)}{G(t)} = \boxed{a(x,t)\,\frac{F''(x)}{F(x)}} \neq \text{const}$$

Step 2 (BCs)

$u(x,t) = F(x)\,G(t)$

$$\begin{cases} 0 = u(0,t) = F(0)\,G(t) \Rightarrow \boxed{F(0)=0} \text{ or } G(t) \not\equiv 0 \ \forall t \ge 0 \\ 0 = u(L,t) = F(L)\,G(t) \Rightarrow \boxed{F(L)=0} \end{cases}$$

$\Rightarrow \boxed{u(x,t) \equiv 0}$

$$\begin{cases} F''(x) - kF(x) = 0 \quad x \in [0,L] \\ F(0) = F(L) = 0 \end{cases} \Rightarrow k = -p^2, \quad p_n = \frac{n\pi}{L}$$

$$\boxed{F_n(x) = \sin \frac{n\pi}{L} x \qquad n = 1, 2, \cdots}$$

$k_n = -\left(\frac{n\pi}{L}\right)^2 = -p_n^2$

$\ddot{G} \boxed{-c^2 k} G = 0 \qquad k = -p_n^2 \qquad -c^2 k = c^2 p_n^2 = \lambda_n^2$

$$\begin{cases} \boxed{\ddot{G}(t) + \lambda_n^2 G(t) = 0} \\ \boxed{\lambda_n = c\,p_n} = \frac{cn\pi}{L} \end{cases} \overset{?}{\Longrightarrow} \boxed{G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t}$$

$0 = s^2 + \lambda_n^2 \Rightarrow s = \pm\sqrt{-\lambda_n^2} = \pm \lambda_n i$

$$u_n(x,t) = F_n(x)\,G_n(t) = \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right) \sin \frac{n\pi}{L} x$$

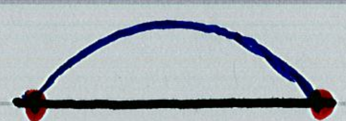
eigenvalues $\lambda_n = \frac{cn\pi}{L}$, $n=1,2,\cdots$. **spectrum** $\{\lambda_1, \lambda_2, \cdots\}$

eigenfunctions $u_n(x,t) = \left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right) \sin\frac{n\pi}{L}x$

n^{th} normal mode $u_n(x,t)$: harmonic motion with frequency $\frac{\lambda_n}{2\pi} = \frac{cn}{2L}$ per unit time

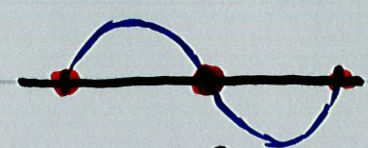
nodes $x_n = \frac{L}{n}, \frac{2L}{n}, \cdots, \frac{n-1}{n}L$ where $u_n(x_n,t)=0 \Leftrightarrow \sin\frac{n\pi}{L}x = 0$

$\frac{n\pi}{L}x = m\pi$

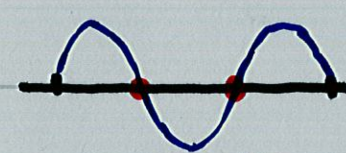


n=1

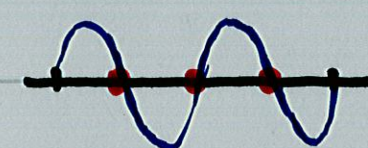
fundamental mode



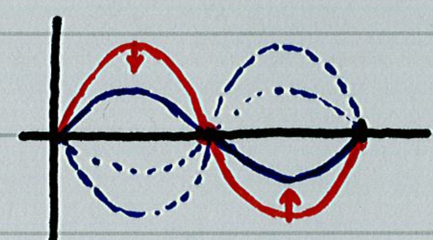
n=2



n=3



n=4



n=2 for various t

Step 3 Solution of the Entire Prob. Fourier Series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t \right) \sin\frac{n\pi}{L}x$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

ICs

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x = f(x) \text{ on } [0,L]$$

$$B_n^* = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$$g(x) = u_t(x,0) = u_t\Big|_{t=0} = \sum_{n=1}^{\infty}\left[-\lambda_n B_n \sin\lambda_n t + \lambda_n B_n^* \cos\lambda_n t\right]\sin\frac{n\pi}{L}x\,\Big|_{t=0}$$

$$= \sum_{n=1}^{\infty} \lambda_n B_n^* \sin\frac{n\pi}{L}x = g(x)$$

$B_n = ?$ $B_n^* = ?$

$$\lambda_n B_n^* = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$$\begin{cases} u_{tt} = c^2 u_{xx} & \text{in } \Omega = [0, L] \times [0, +\infty) \\ u(0,t) = u(L,t) = 0 & \forall\, t \in [0, +\infty) \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & \forall\, x \in [0, L] \end{cases}$$

<u>IB-value prob.</u>

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos \frac{cn\pi}{L} t + B_n^* \sin \frac{cn\pi}{L} t \right) \sin \frac{n\pi}{L} x$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x$$

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

Solution (12) Established.

let $g(x) \equiv 0$ for simplicity

$f(x)$ on $[0, L]$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\frac{cn\pi}{L}t \sin\frac{n\pi}{L}x \quad , \quad B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

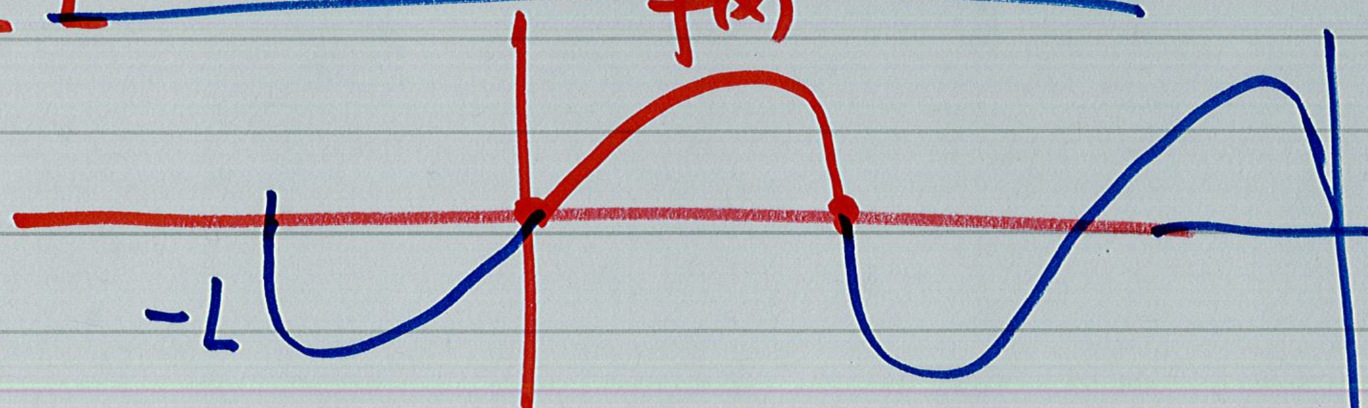
$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) + \sin(B-A)\right]$$

$$u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right]$$

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}(x+ct) + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}(x-ct)\right]$$

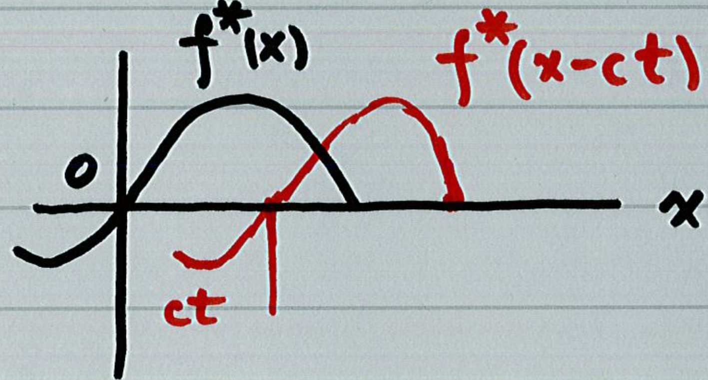
$$= \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right]$$

$f^* = ?$



Physical Interpretation

(17) $$u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct) \right]$$



traveling wave

Ex. 1 $$f(x) = \begin{cases} \frac{2k}{L}x & 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \frac{L}{2} < x < L \end{cases}, \quad g(x) = 0$$

$$u(x,t) = \frac{8k}{\pi^2}\left[\sin\frac{\pi x}{L}\cos\frac{c\pi t}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L}\cos\frac{3c\pi t}{L} + \cdots \right]$$

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