§12.4 D'Alemberts Solution of the Wave Eq. Characteristics.

$u_{tt} = c^2 u_{xx}$ with $c^2 = T/\rho$.

$$\begin{cases} v = x + ct \\ w = x - ct \end{cases} \Longrightarrow \frac{\partial^2 u}{\partial w \partial v} = 0$$

$$\frac{\partial^2 u}{\partial w \partial v} = 0 \implies \frac{\partial u}{\partial v} = h(v) \implies u(v,w) = \int h(v)\,dv + \psi(w)$$

$$\begin{cases} v = x + ct \\ w = x - ct \end{cases} \qquad = \varphi(v) + \psi(w)$$

$$\implies u(x,t) = \varphi(v) + \psi(w)$$

$$= \varphi(x+ct) + \psi(x-ct) \qquad \text{d'Alemberts' solution}$$

<u>ICs</u> $\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \implies u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

Characteristic. Types and Normal Forms of PDEs.

quasilinear $A\,u_{xx} + 2B\,u_{xy} + C\,u_{yy} = F(x, y, u, u_x, u_y)$

- Hyperbolic: $AC - B^2 < 0$ $\quad u_{tt} - c^2 u_{xx} = 0 \quad \xRightarrow{y = ct} \quad c^2(u_{yy} - u_{xx}) = 0$
- Parabolic: $AC - B^2 = 0$ $\quad u_t - c^2 u_{xx} = 0 \quad \xRightarrow{y = c^2 t} \quad c^2(u_y - u_{xx}) = 0$
- Elliptic: $AC - B^2 > 0$ $\quad u_{xx} + u_{yy} = 0$

characteristic eq. $A\,y'^2 - 2B\,y' + C = 0 \qquad y' = \dfrac{dy}{dx}$

Ex. 1 $u_{tt} - c^2 u_{xx} = 0 \xrightarrow{y = ct} u_{xx} - u_{yy} = 0 \qquad A = 1,\ B = 0,\ C = -1$

characteristic eq. $0 = (y')^2 - 1 = (y'+1)(y'-1)$

$$\Longrightarrow \begin{cases} y' + 1 = 0 \\ y' - 1 = 0 \end{cases} \Longrightarrow \begin{cases} \text{const} = y + x = \varphi(x,y) \\ \text{const} = y - x = \psi(x,y) \end{cases}$$

new variables $\begin{cases} v = \varphi(x,y) = y + x = ct + x \\ w = \psi(x,y) = y - x = ct - x \end{cases} \Longrightarrow u_{vw} = 0 \Longrightarrow u(x,t) = f_1(x+ct) + f_2(x-ct)$

Type	New Variables	Normal Forms
Hyperbolic	$v = \varphi(x,y),\ w = \psi(x,y)$	$u_{vw} = F_1$
Parabolic	$v = x,\ w = \varphi(x,y) = \psi(x,y)$	$u_{ww} = F_2$
Elliptic	$v = \frac{1}{2}(\varphi + \psi),\ w = \frac{1}{2i}(\varphi - \psi)$	$u_{vv} + u_{ww} = F_3$

#5 on p556

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

$c=1$, $k=0.01$, $L=1$

$f(x) = k\sin\pi x \implies u(x,t) = \frac{k}{2}\left[\sin(x+t)\pi + \sin(x-t)\pi\right]$

$$= \frac{k}{2}\left[\sin \pi x \cos t\pi + \cos\pi x \sin \pi t + \sin\pi x \cos\pi t - \cos\pi x \sin\pi t\right]$$

$$= k \sin\pi x \cos\pi t$$

#19 on p556

$$u_{tt} = c^2 u_{xx} \quad \text{with} \quad c^2 = \frac{E}{\rho}$$

BCs $\quad u(0,t) = 0, \quad u_x(L,t) = 0$

ICs $\quad u(x,0) = f(x), \quad u_t(x,0) = 0$

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin p_n x \cos p_n c t$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin p_n x \, dx, \quad p_n = \frac{(2n+1)\pi}{2L}$$