§12.4 D'Alembert's Solution of the Wave Eq. Characteristics.

$u_{tt} = c^2 u_{xx}$ with $c^2 = T/\rho$.

$u(x,t) = u(v,w)$

$$\begin{cases} v = x + ct \\ w = x - ct \end{cases} \Longrightarrow \boxed{\frac{\partial^2 u}{\partial w \partial v} = 0}$$

$u_x = u_v + u_w$, $c u_{xx} = (u_{vv} + u_{vw}) + u_{wv} + u_{ww}) c^2$

$= \boxed{(u_{vv} + u_{ww}) + 2u_{vw}} c^2$

$$\boxed{4c^2 u_{vw} = 0}$$

$$\frac{\partial u}{\partial v} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial v}$$

$$\boxed{\frac{\partial u}{\partial t}_x = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial t}_x + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial t}_x} = c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial w}$$

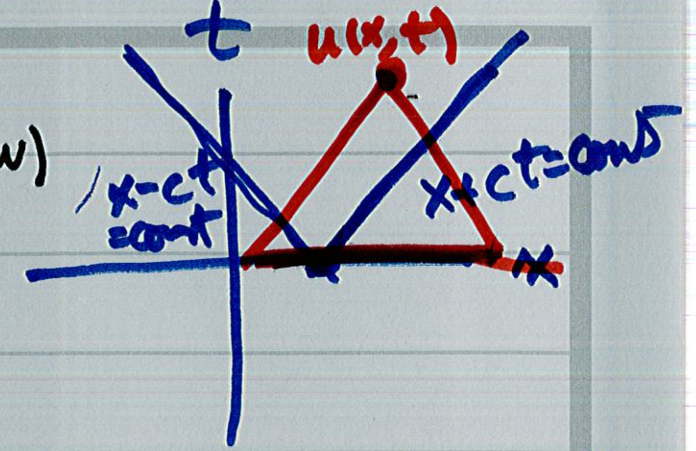
$$u_{tt} = \frac{\partial}{\partial t}\left[c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial w}\right] = c\left[\frac{\partial^2 u}{\partial v^2} \cdot \frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial w \partial v} \cdot \frac{\partial w}{\partial t}\right]$$

$$- c\left[\frac{\partial^2 u}{\partial v \partial w} \cdot \frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial w^2} \cdot \frac{\partial w}{\partial t}\right]$$

$$= c^2\left[u_{vv} - u_{vw}\right] - c^2\left[u_{wv} - u_{ww}\right] = c^2(u_{vv} + u_{ww}) - 2c^2 u_{vw}$$

$$\int \frac{\partial^2 u}{\partial w \partial v}\, dw = 0 \Longrightarrow \frac{\partial u}{\partial v} = h(v) \Longrightarrow u(v,w) = \int h(v)\, dv + \psi(w)$$

$$\begin{cases} v = x+ct \\ w = x-ct \end{cases} \quad \frac{\partial u}{\partial v} = \int \frac{\partial^2 u}{\partial w \partial v}\, dw = \text{const} = h(v) \qquad = \underline{\varphi}(v) + \underline{\psi}(w)$$



$$\Longrightarrow u(x,t) = \varphi(v) + \psi(w)$$

$$= \underline{\varphi}(x+ct) + \underline{\psi}(x-ct)$$

$\boxed{\text{d'Alembert's solution}}$

$x+ct = \text{const}$

$x-ct = \text{const}$

$\underline{\text{ICs}}$ $\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \Longrightarrow \boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds}$

$$\underline{f(x) = u(x,0) = \varphi(x) + \psi(x)} \qquad \varphi' - \psi' = \frac{1}{c} g(x)$$

$$u_t\big|_{t=0} = \varphi'(x)\cdot c + \psi'(x)(-c) \Rightarrow \underline{\varphi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(s)\, ds} + K_0 \ \cancel{+\varphi(x_0) - \psi(x_0)}$$

$$= c\left[\varphi'(x) - \psi'(x)\right] = g(x) \qquad \varphi = \frac{1}{2} f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\, ds + \frac{1}{2}K_0$$

$$\varphi(x) + \psi(x) = f(x)$$

$$\varphi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(s)\,ds + K_0$$

$$\Rightarrow \varphi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + K_0/2$$

$$\Rightarrow \psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds - K_0/2$$

$$u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

$$= \boxed{\frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct)} + \underline{\frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds} + \cancel{\frac{K_0}{2}} - \frac{1}{2c}\int_{x_0}^{x-ct} g(s)\,ds - \cancel{\frac{K_0}{2}}$$

$$\underline{\frac{1}{2c}\int_{x-ct}^{x_0} g(s)\,ds}$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

Characteristic, Types and Normal Forms of PDEs.

$B=0$ $A=-1$, $C=1$

quasilinear $\quad A\,u_{xx} + 2B\,u_{xy} + C\,u_{yy} = F(x, y, u, u_x, u_y)$

- Hyperbolic: $\quad AC - B^2 < 0 \quad u_{tt} - c^2 u_{xx} = 0 \xrightarrow{y=ct} c^2(u_{yy} - u_{xx}) = 0$
- Parabolic: $\quad AC - B^2 = 0 \quad u_t - c^2 u_{xx} = 0 \xrightarrow{y=c^2 t} c^2(u_y - u_{xx}) = 0$
- Elliptic: $\quad AC - B^2 > 0 \quad u_{xx} + u_{yy} = 0$

$A=-1$, $B=0$, $C=0$

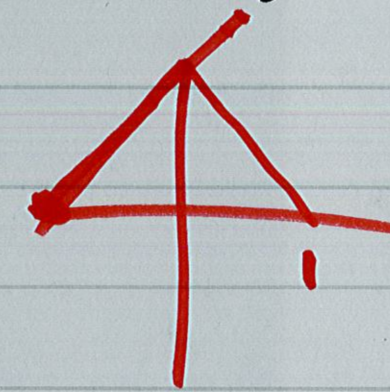
$A=1$, $B=0$, $C=1$

characteristic eq. $\quad A(y')^2 - 2B\,y' + C = 0 \qquad y' = \dfrac{dy}{dx}$

Ex. 1 $u_{tt} - c^2 u_{xx} = 0 \xrightarrow[]{y = ct} u_{xx} - u_{yy} = 0$ $A = 1,\ B = 0,\ C = -1$

characteristic eq. $0 = (y')^2 - 1 = (y'+1)(y'-1)$

$$\Longrightarrow \begin{cases} y' + 1 = 0 \\ y' - 1 = 0 \end{cases} \Longrightarrow \begin{cases} \text{const} = y + x = \varphi(x,y) \\ \text{const} = y - x = \psi(x,y) \end{cases}$$



new variables $\begin{cases} v = \varphi(x,y) = y + x = ct + x \\ w = \psi(x,y) = y - x = ct - x \end{cases} \Longrightarrow u_{vw} = 0 \Longrightarrow u(x,t) = f_1(x+ct) + f_2(x-ct)$

normal form

Type	New Variables	Normal Forms
Hyperbolic	$v=\varphi(x,y),\ w=\psi(x,y)$	$u_{vw}=F_1$
Parabolic	$v=x,\ w=\varphi(x,y)=\psi(x,y)$	$u_{ww}=F_2$
Elliptic	$v=\frac{1}{2}(\varphi+\psi),\ w=\frac{1}{2i}(\varphi-\psi)$	$u_{vv}+u_{ww}=F_3$

#19 on P556

$$u_{tt} = c^2 u_{xx} \quad \text{with} \quad c^2 = \frac{E}{\rho}$$

BCs $u(0,t) = 0, \quad u_x(L,t) = 0$

ICs $u(x,0) = f(x), \quad u_t(x,0) = 0$

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin p_n x \cos p_n c t$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin p_n x \, dx, \quad p_n = \frac{(2n+1)\pi}{2L}$$

#5 on p556 $\quad u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$

$c=1$, $k=0.01$, $L=1$

$f(x) = k\sin\pi x \implies u(x,t) = \frac{k}{2}\left[\sin(x+t)\pi + \sin(x-t)\pi\right]$

$$= \frac{k}{2}\left[\sin\pi x\cos t\pi + \cos\pi x\sin\pi t + \sin\pi x\cos\pi t - \cos\pi x\sin\pi t\right]$$

$$= k\sin\pi x\cos\pi t$$