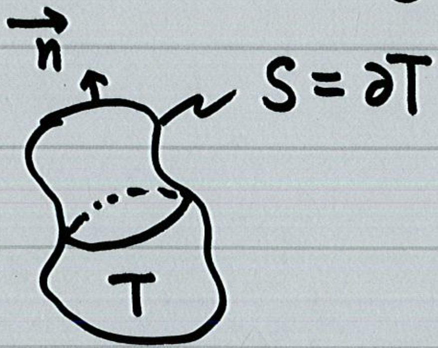
§12.5 Modeling: Heat Flow from a Body in Space. Heat Equation



$\vec{v}$ — velocity of the heat flow

$\iint_S \vec{v}\cdot\vec{n}\,dS$ — the total amount of heat that flows across S from T

σ — specific heat
ρ — density
u — temperature

$H = \iiint_T \sigma \rho u \, dV$ — the total amount of heat in T

$-\dfrac{\partial H}{\partial t} = -\iiint_T \sigma \rho \dfrac{\partial u}{\partial t}\, dV$ — the time rate of decrease of H

$$\Longrightarrow \quad -\iiint_T \sigma \rho \frac{\partial u}{\partial t}\, dV = \iint_S \vec{v}\cdot\vec{n}\, dS$$

Fourier's Law

$$\vec{V} = -K \nabla u \qquad K \text{ — thermal conductivity}$$

$$-\iiint_T \sigma \rho \frac{\partial u}{\partial t}\, dV = \iint_S \vec{V} \cdot \vec{n}\, dS$$

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u$$

§12.6 Heat Eq: Solution by F-series. Steady 2D Heat Eq. Dirichlet Problem.

1D Heat Eq. $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ $\quad (x,t) \in [0,L] \times [0,+\infty)$

BCs $\quad u(0,t)=0, \quad u(L,t)=0 \quad \forall\, t \geq 0$

ICs $\quad u(x,0) = f(x) \quad \forall\, x \in [0,L]$

Step 1 (Separation of Variables)

$$u(x,t) = F(x)\, G(t)$$

Step 2 (BCs)

$$\begin{cases} F''(x) + p^2 F(x) = 0 \\ F(0) = F(L) = 0 \end{cases}$$

$$\dot{G}(t) + \lambda^2 G(t) = 0$$

Step 3 (Final Solution. F-Series)

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t}, \qquad \lambda_n = \frac{cn\pi}{L}$$

$f(x) = u(x,0)$

Ex. 1 $\quad L = 80 \text{ cm}, \quad f(x) = 100 \sin \frac{\pi x}{80} \ ^\circ C$

$$u(0,t) = u(80,t) = 0 \ ^\circ C, \quad \rho = 8.92 \ g/cm^3, \quad \sigma = 0.092 \ cal/(g \ ^\circ C)$$

$$K = 0.95 \ cal/(cm \ sec \ ^\circ C)$$

Find t such that $\max\limits_{0 \le x \le L} u(x,t) = 50 \ ^\circ C$

Solution $\quad c^2 = K/(\sigma\rho) = 1.158 \ cm^2/sec$

$$100 \sin \frac{\pi x}{80} = u(0,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{80} \Longrightarrow \begin{array}{l} B_1 = \\ B_2 = \\ \vdots \end{array}$$

Ex. 2 $\quad f(x) = u(x,0) = 100 \sin \frac{3\pi x}{80}$, $\lambda_3^2 = 0.01607$

$$\Rightarrow \quad u(x,t) = 100 \sin \frac{3\pi x}{80} e^{-0.01607 t}$$

$$50^\circ C = \max_{0 \le x \le 80} u(x,t) \quad \Longrightarrow \quad t = \frac{\ln 0.5}{-0.01607} \approx 43 \text{ sec.}$$

Steady 2D Heat Problems. Laplace Eqs.

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u$$

steady $\frac{\partial u}{\partial t} = 0 \quad \Longrightarrow \quad \nabla^2 u = 0 \quad \text{in } \Omega$

BCs
- Dirichlet $\quad u\big|_{\partial\Omega} = f$
- Neumann $\quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g$
- Robin $\quad \frac{\partial u}{\partial n}\Big|_{\Gamma_N} = g, \; u\big|_{\Gamma_D} = f$

Dirichlet Problem in a Rectangle R