p552 #15, #16

$$c^2 = \frac{EI}{\rho A}$$

(21) $u_{tt} = -c^2 u_{xxxx}$

#15 $u(x,t) = F(x)\,G(t)$

$u_t = F\dot{G}$, $\quad u_x = F'G$, $\quad u_{xxxx} = F^{(4)}(x)\,G(t)$

$$u_{tt} = \frac{F(x)\ddot{G}(t) = -c^2 F^{(4)}(x)\,G(t)}{-c^2 F(x)\,G(t)}$$

$\forall (x,t)$

$$\frac{F^{(4)}(x)}{F(x)} = \frac{\ddot{G}(t)}{-c^2 G(t)} = \underset{\substack{\parallel \\ C}}{\text{const}} = \beta^4$$

$F^{(4)}(x) - C\,F(x) = 0$

$\ddot{G}(t) + c^2\,C\,G(t) = 0$

$\boxed{F^{(4)}(x) - \beta^4 F(x) = 0}$

$\boxed{\ddot{G}(t) + c^2\beta^4 G(t) = 0}$

$F(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$

$\cosh\beta x = \dfrac{e^{\beta x} + e^{-\beta x}}{2}$ $\qquad$ $\sinh\beta x = \dfrac{e^{\beta x} - e^{-\beta x}}{2}$

$0 = s^4 - \beta^4 = (s^2+\beta^2)(s^2-\beta^2)$

$= (s^2+\beta^2)(s+\beta)(s-\beta)$

$a^2 - b^2 = (a+b)(a-b)$

$a = s^2$, $b = \beta^2$

$s_1 = -\beta$, $s_2 = \beta$ $\qquad$ $s_{3,4} = \pm i\beta$

$e^{-\beta x}$, $e^{\beta x}$ $\qquad$ $\cos\beta x$ $\qquad$ $\sin\beta x$

$$\ddot{G}(t) + \underline{\underline{c^2 p^4}}\, G(t) = 0$$

$$0 = s^2 + c^2 p^4 \implies s_{1,2} = \pm \underline{c p^2}\, i$$

#16 **BCs** $u_{tt} = -c^2 \underline{u_{xxxx}}$ $x \in [0, L]$, $t \in [0, +\infty)$

$$\boxed{\begin{matrix} u(0,t) = 0, & u(L,t) = 0 & \forall\, t \geq 0 \\ u_{xx}(0,t) = 0, & u_{xx}(L,t) = 0 & \forall\, t \geq 0 \end{matrix}}$$

$$\boxed{\begin{matrix} u_x(0,t) = 0 \\ u_x(L,t) = 0 \end{matrix}}$$

ICs $u_t(x,0) = 0$, $u(x,0) = f(x)$ $\forall\, x \in [0, L]$

$F_n(x)\, G_n(t)$

$$0 = u(0,t) = F(0)\,G(t) \Longrightarrow F(0) = 0 = A + C$$

$$\boxed{F(L) = 0} =$$

$$(\cosh x)' = \sinh x$$

$$(\sinh x)' = \cosh x$$

$$0 = u_{xx}(0,t) = F''(0)\,G(t) \Longrightarrow F''(0) = 0$$

$$\boxed{F''(L) = 0}$$

$$F(x) = \underline{A}\cos\beta x + \underline{B}\sin\beta x + \underline{C}\cosh\beta x + \underline{D}\sinh\beta x$$

$$\cosh\beta x = \frac{e^{\beta x} + e^{-\beta x}}{2} \qquad \sinh\beta x = \frac{1}{2}\left(e^{\beta x} - e^{-\beta x}\right)$$

$$F' = -A\beta\sin\beta x + \beta B\cos\beta x$$

$$+ C\beta\sinh\beta x + D\beta\cosh\beta x$$

$$F'' = -A\beta^2\cos\beta x - B\beta^2\sin\beta x + C\beta^2\cosh x + D\beta^2\sinh\beta x$$

$$0 = F''(0) = -A\cancel{\beta^2} + C\cancel{\beta^2} \Longrightarrow A = 0 = C$$

$$F(L) = 0 = B \sin \beta L + D \sinh \beta L \qquad e^{\beta L} - e^{-\beta L} \neq 0$$

$$F''(L) = 0 = -B\cancel{\beta^2} \sin \beta L + D\cancel{\beta^2} \sinh \beta \cancel{L}$$

$$\begin{bmatrix} \sin\beta L & \sinh\beta L \\ -\sin\beta L & \sinh\beta L \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \boxed{\beta_n = \frac{n\pi}{L} \quad n = 1, 2, \dots}$$

$$2D \sinh \beta L = 0 \implies D = 0 \qquad F_n(x) = \sin \frac{n\pi}{L} x.$$

$$\cancel{\sin \beta L \neq 0}$$

$$B \boxed{\sin \beta L} = 0$$

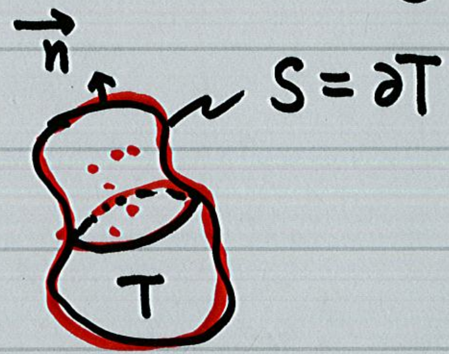
$$\sin \beta L = 0 \implies \beta L = n\pi \quad n = 1, 2, \dots$$

$$\begin{cases} F^{(4)}(x) - \beta^4 F(x) = 0 \\ F(0) = F(L) = 0 \\ F''(0) = F''(L) = 0 \end{cases}$$

$\beta_n = \frac{n\pi}{L}$ for $n = 1, 2, \cdots$

$F_n(x) = \sin \frac{n\pi x}{L}$

§12.5 Modeling: Heat Flow from a Body in Space. Heat Equation



$\vec{v}$ — velocity of the heat flow

$\displaystyle\iint_S \vec{v}\cdot\vec{n}\,dS$ — the total amount of heat that flows across S from T

σ — specific heat

ρ — density

u — temperature

$\displaystyle H = \iiint_T \sigma\rho u\,dV$ — the total amount of heat in T

$\displaystyle -\frac{\partial H}{\partial t} = -\iiint_T \sigma\rho\frac{\partial u}{\partial t}\,dV$ — the time rate of decrease of H

$$\Longrightarrow \quad -\iiint_T \sigma\rho\frac{\partial u}{\partial t}\,dV = \iint_S \vec{v}\cdot\vec{n}\,dS$$

Fourier's Law

$$\vec{v} = -K \nabla u \qquad \nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix}$$

$$\iiint_T \operatorname{div} \vec{v} \, dV = \iint_{\partial T} \vec{v} \cdot \vec{n} \, ds$$

K — thermal conductivity

$$\operatorname{div} \vec{v} = \operatorname{div}(v_1, v_2, v_3) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$-\iiint_T \sigma \rho \frac{\partial u}{\partial t} \, dV = \iint_S \vec{v} \cdot \vec{n} \, ds = \iint_{S = \partial T} (-K \nabla u) \cdot \vec{n} \, ds$$

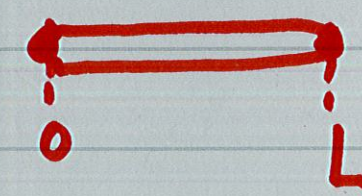
$$= -\iiint_T \operatorname{div}(K \nabla u) \, dV = -\iiint_T K \operatorname{div} \nabla u \, dV$$

$$\nabla \cdot \nabla = \operatorname{div} \nabla = (\partial_x, \partial_y, \partial_z) \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

$$\boxed{\frac{\partial u}{\partial t} = c^2 \nabla^2 u} = c^2 (u_{xx} + u_{yy} + u_{zz}) \qquad \nabla^2 = \Delta = \partial_{xx} + \partial_{yy} + \partial_{zz}$$

§12.6 Heat Eq: Solution by F-series. Steady 2D Heat Eq. Dirichlet Problem.

$u(x,t)$



1D Heat Eq. $\quad \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} \qquad (x,t) \in [0,L] \times [0,+\infty)$

BCs $\quad u(0,t) = 0, \; u(L,t) = 0 \qquad \forall\, t \geq 0$

ICs $\quad u(x,0) = f(x) \qquad \forall\, x \in [0,L]$

Step 1 (Separation of Variables)

$$u(x,t) = \underline{F}(x)\,\underline{G}(t)$$

Step 2 (BCs)

$$\begin{cases} F''(x) + p^2 F(x) = 0 \\ F(0) = F(L) = 0 \end{cases}$$

$$\dot{G}(t) + \lambda^2 G(t) = 0 \qquad \lambda_n^2 = c^2 p_n^2$$

$$\begin{cases} p_n = \dfrac{n\pi}{L} \quad \text{for } n = 1, 2, \cdots \\ F_n(x) = \sin \dfrac{n\pi x}{L} \end{cases}$$

$$0 = u(0,t) = F(0)\, G(t)$$

$$0 = u(L,t) = F(L)\, G(t)$$

$$0 = \dot{G}_n(t) + \lambda_n^2 G_n(t)$$

$$0 = s^2 + p^2 \Longrightarrow s = \pm p i$$

$$G_n(t) = e^{-\lambda_n^2 t} = e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

$$F(x) = A \cos px + B \sin px$$

$$0 = F(0) = A$$

$$0 = F(L) = \underline{B} \sin pL \Longrightarrow \sin pL = 0 \Rightarrow pL = n\pi$$

$$n = 1, 2, \cdots$$

- $u(x,t) = \underline{\underline{F(x)}}\, G(t)$

$$\frac{u_t = F\dot{G} = c^2 u_{xx} = c^2 F'' G}{c^2 F(x)\, G(t)}$$

$$\frac{\dot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = \text{const} = \underline{\underline{-p^2}} = D$$

$$\begin{cases} F''(x) + p^2 F(x) = 0 \\ \dot{G}(t) + c^2 p^2 G(t) = 0 \end{cases}$$

$$\underline{F'' - DF = 0}$$

$D = 0 \longrightarrow F = c_1 + c_2 x$

$\underline{\underline{D = -\nu^2}}$

$D = \nu^2 \longrightarrow F = c_1 e^{\nu t} + c_2 e^{-\nu t}$

Step 3 (Final Solution. F-Series)

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, \boxed{e^{-\lambda_n^2 t}}}$$

$$\lambda_n = \frac{cn\pi}{L}$$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \implies \boxed{B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx}$$

Wave Eq

$$u(x,t) = \sum \left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right)$$

Ex. 1 $L = 80\text{ cm}, \quad f(x) = 100 \sin\frac{\pi x}{80} \ {}^\circ C$

$u(0,t) = u(80,t) = 0\ {}^\circ C, \quad \rho = 8.92\ g/cm^3, \quad \sigma = 0.092\ cal/(g\ {}^\circ C),$

$e^{-\frac{1.158\pi^2}{80^2}t} = 50\ {}^\circ C \qquad K = 0.95\ cal/(cm\ sec\ {}^\circ C)$

$t \approx 6.5\text{ min}$

Find t such that $\max\limits_{0 \le x \le L} u(x,t) = 50\ {}^\circ C$

Solution $c^2 = K/(\sigma\rho) = 1.158\ cm^2/sec$

$$\underbrace{100 \sin\frac{\pi x}{80}} = u(0,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{80} \Longrightarrow \begin{array}{l} B_1 = 100 \\ B_2 = 0 \\ \vdots \end{array}$$

$$B_1 \sin\frac{\pi x}{80} + B_2 \sin\frac{2\pi x}{80} + B_3 \sin\frac{3\pi x}{80} + \cdots$$

$$B_n = \frac{2}{80}\int_0^{80} 100 \sin\frac{\pi x}{80} \cdot \sin\frac{n\pi x}{80}\,dx$$

$$u(x,t) = 100 \boxed{\sin\frac{\pi x}{80}}\, e^{-\left(\frac{\sqrt{1.158}\,\pi}{80}\right)^2 t}$$

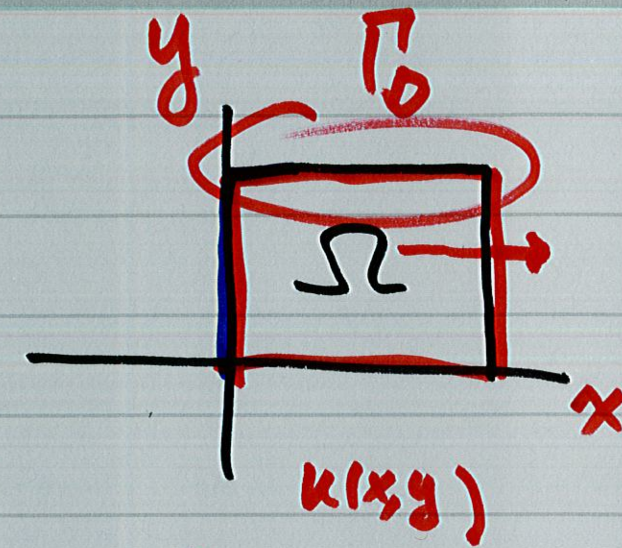
Ex. 2 $\quad f(x) = u(x,0) = 100 \sin \dfrac{3\pi x}{80}$, $\lambda_3^2 = 0.01607$ $\quad B_n = 0$, $B_3 = 100$

$$\Rightarrow \quad u(x,t) = 100 \sin \frac{3\pi x}{80} e^{-0.01607t}$$

$$50^\circ C = \max_{0 \le x \le 80} u(x,t) \quad \Longrightarrow \quad t = \frac{\ln 0.5}{-0.01607} \approx \underline{\underline{43 \text{ sec}}}.$$

$$\max_{0 \le x \le 80} u(x,t) = 100 e^{-0.01607t}$$

Steady 2D Heat Problems. Laplace Eqs.



$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

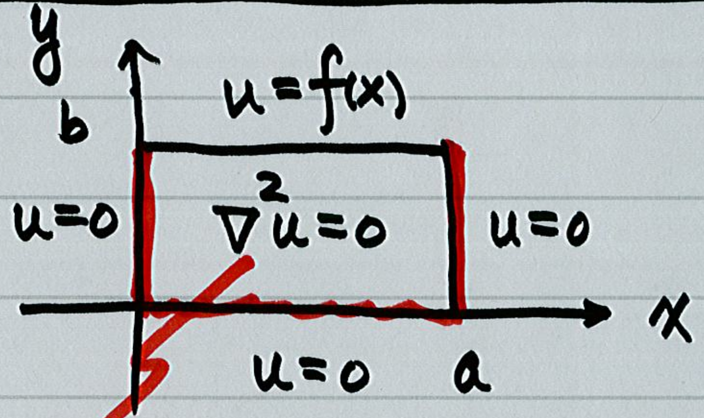
$\boxed{\text{steady} \quad \frac{\partial u}{\partial t} = 0}$ $\Longrightarrow$ $\boxed{\nabla^2 u = 0 \quad \text{in } \Omega}$

$$\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n}$$

BCs

- Dirichlet $\quad u\big|_{\partial\Omega} = f$
- Neumann $\quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g$
- Robin $\quad \frac{\partial u}{\partial n}\Big|_{\Gamma_N} = g, \; u\big|_{\Gamma_D} = f$

Dirichlet Problem in a Rectangle R



$$u(x,y) = F(x)\,G(y)$$

$$F''(x)G(y) + F(x)G''(y) = 0$$

$$\frac{F''(x)\,G(y) = -F(x)\,G''(y)}{F(x)\,G(y)}$$

$u_{xx} + u_{yy}$

$$\boxed{\frac{F''(x)}{F(x)}} = -\frac{G''(y)}{G(y)} = -k^2$$

$$\boxed{\begin{array}{c} F''(x) + k^2 F(x) = 0 \\ F(0)=0,\ F(a)=0 \end{array}}$$

$s = \pm k$

$$c_1 e^{kx} + c_2 e^{-kx}$$

$s = \pm ik$

$$F(x) = c_1 \cos kx + c_2 \sin kx$$

$$0 = F(a) = c_2 \sin ka \Rightarrow \boxed{k_n = \frac{n\pi}{a}}$$

$ka = n\pi$

$$\boxed{F_n(x) \sin \frac{n\pi x}{a}}$$

$$G''(y) - k^2 G(y) = 0 \qquad s^2 - k^2 = 0 \Rightarrow s = \pm k$$

$$G(y) = c_1 e^{ky} + c_2 e^{-ky} \qquad k_n = \frac{n\pi}{a}$$

$$G_n(y) = c_1 e^{k_n y} + c_2 e^{-k_n y} = 2c_1 \left(\frac{e^{k_n y} - e^{-k_n y}}{2} \right)$$

$$0 = u(x,0) = F(x)\,G(0) \Rightarrow G(0) = 0$$

$$\underline{0 = G_n(0)} = c_1 + c_2 \Rightarrow c_2 = -c_1$$

$$\underline{G_n(y)} = \sinh(k_n y) \qquad F_n(x) = \sin\frac{n\pi x}{a}$$

$$u_n(x,y) = A_n \sin\frac{n\pi x}{a} \sinh\left(\frac{n\pi y}{a}\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} \underline{\underline{A_n}} \sin\frac{n\pi x}{a} \sinh\left(\frac{n\pi y}{a}\right)$$

$$f(x) = u(x,b) = \sum_{n=1}^{\infty} \left(\underline{\underline{A_n \sinh\left(\frac{n\pi b}{a}\right)}}\right) \sin\frac{n\pi x}{a}$$

$$A_n = \frac{1}{\sinh\left(\frac{n\pi b}{a}\right)} \frac{2}{a} \int_0^a f(x) \sin\frac{n\pi x}{a}\, dx$$