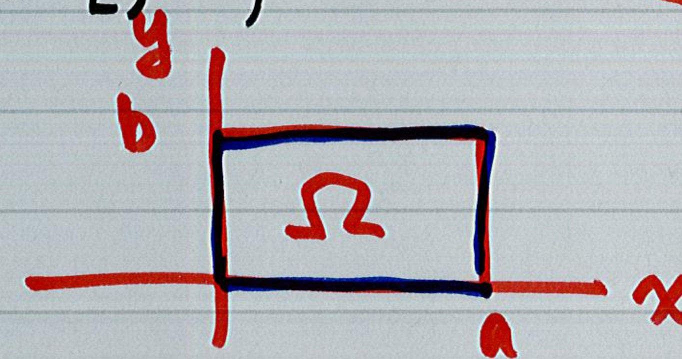
§12.9 Rectangular Membrane. Double Fourier Series.

$u(x,t)$, $u(x,y)$

$\boxed{u(x,y,t)}$

$$\begin{cases} u_{tt} = c^2\left(u_{xx} + u_{yy}\right) & (x,y) \in [0,a] \times [0,b] = \Omega,\ t \in [0,+\infty) \\ \underline{\text{BCs}} & u\big|_{\partial\Omega} = 0 \\ \underline{\text{ICs}} & u(x,y,0) = f(x,y),\ u_t(x,y,0) = g(x,y) \end{cases}$$



<u>Step 1</u> $\quad u(x,y,t) = \underline{F(x,y)}\, G(t)$

$$\quad\quad\quad\quad\quad\quad \| $$

$$\quad\quad\quad\quad\quad\quad \underline{H(x)Q(y)}\, G(t)$$

$$\frac{\ddot{G}(t)}{c^2 G(t)} = \boxed{\frac{F_{xx} + F_{yy}}{F(x,y)} = -\nu^2}$$

$$\frac{u_{tt} = F\ddot{G} = c^2\left(F_{xx} + F_{yy}\right) G}{c^2 G(t)\, F(x,y)}$$

$$\begin{cases} \ddot{G}(t) + \lambda^2 G(t) = 0 & \text{with } \lambda = c\nu \quad \text{ODE} \\ \boxed{F_{xx} + F_{yy} + \nu^2 F = 0} & \text{Helmholt equation} \quad \text{PDE} \quad F(x,y) \end{cases}$$

$$F(x,y) = H(x)\,Q(y)$$

$$\boxed{\begin{cases} H''(x) + k^2 H(x) = 0 \\ Q''(y) + p^2 Q(y) = 0 \end{cases}} \quad \text{with } p^2 = \nu^2 - k^2$$

$$\frac{H''Q + HQ'' + \nu^2 HQ = 0}{HQ}$$

$$\frac{H''}{H} + \frac{Q''}{Q} + \nu^2 = 0$$

$$Q'' = (k^2 - \nu^2)\,Q$$

$$\frac{Q''(y)}{Q(y)} + \nu^2 = \boxed{-\frac{H''(x)}{H(x)} = k^2}$$

$$\begin{cases} \ddot{G}(t) + \lambda^2 G(t) = 0, & \lambda = c\nu \\ H''(x) + k^2 H(x) = 0 \\ Q''(y) + p^2 Q(y) = 0, & p_n^2 = \nu_{mn}^2 - k_m^2 \end{cases}$$

$$u = F(x,y)\,G(t) = H(x)\,Q(y)\,G(t)$$

$$\boxed{\sin\frac{m\pi x}{a} = H_m(x) \qquad k_m = \frac{m\pi}{a} \quad m = 1, 2, \cdots}$$

$$\boxed{p_n = \frac{n\pi}{a} \quad n = 1, 2, \cdots \qquad Q_n(y) = \sin\frac{n\pi y}{b}}$$

Step 2 (BCs)

$$0 = u(0,y,t) = H(0)\,\underline{Q(y)G(t)} \Rightarrow$$

$$0 = u(a,y,t)$$

$$0 = u(x,0,t)$$

$$0 = u(x,b,t)$$

$$\boxed{\begin{array}{l} H(0) = 0 \\ H(a) = 0 \\ \hline Q(0) = 0 \\ Q(b) = 0 \end{array}}$$

$$k_m = \frac{m\pi}{a}, \quad m = 1, 2, \cdots \qquad H_m(x) = \sin\frac{m\pi x}{a}$$

$$p_n = \frac{n\pi}{b}, \quad n = 1, 2, \cdots \qquad Q_n(y) = \sin\frac{n\pi y}{b}$$

$$F_{mn}(x,y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$\ddot{G}_{mn}(t) + \lambda_{mn}^2 G_{mn}(t) = 0 \quad \text{with } \lambda_{mn} = c\nu_{mn}, \quad p_n^2 = \nu_{mn}^2 - k_m^2$$

$$\boxed{\lambda_{mn}^2 = c^2\nu_{mn}^2 = c^2(p_n^2 + k_m^2)}$$

$$G_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B_{mn}^*\sin\lambda_{mn}t$$

$$\lambda_{mn} = c\sqrt{p_n^2 + k_m^2}$$

$$u_{mn}(x,y,t) = F_{mn}(x,y)\,G_{mn}(t)$$

$$0 = s^2 + \lambda_{mn}^2$$

$$\boxed{s = \pm\lambda_{mn} i}$$

step 3

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \overbrace{\left(B_{mn}\cos\lambda_{mn}t + B^{*}_{mn}\sin\lambda_{mn}t\right)}^{G_{mn}(t)} \overbrace{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}^{F_{mn}(x,y)}$$

$$f(x,y) = u(x,y,0) = \sum_{m=1}^{\infty}\left(\underbrace{\sum_{n=1}^{\infty} B_{mn}\sin\frac{n\pi y}{b}}_{=}\right)\sin\frac{m\pi x}{a} = \sum_{n=1}^{\infty} K_m(y)\sin\frac{m\pi x}{a}$$

$$K_m(y) = \frac{2}{a}\int_0^a f(x,y)\sin\frac{m\pi x}{a}\,dx$$

$$\boxed{K_m(y) = \sum_{n=1}^{\infty} B_{mn}\sin\frac{n\pi y}{b}}$$

$$B_{mn} = \frac{2}{b}\int_0^b \boxed{K_m(y)}\sin\frac{n\pi y}{b}\,dy$$

$$= \frac{2}{b}\int_0^b \frac{2}{a}\int_0^a f(x,y)\sin\frac{m\pi x}{a}\,dx\,\sin\frac{n\pi y}{b}\,dy$$

$$= \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\,d\,\sin\frac{n\pi y}{b}\,dx\,dy$$

$$g(x,y) = u_t(x,y,0)$$

$$B^{*}_{mn} = \cdots$$

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(B_{mn}\cos\lambda_{mn}t+B_{mn}^{*}\sin\lambda_{mn}t\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$B_{mn}=\frac{2}{b}\int_0^b K_m(y)\sin\frac{n\pi y}{b}\,dy$$

$$K_m(y)=\frac{2}{a}\int_0^a f(x,y)\sin\frac{m\pi x}{a}\,dx$$

$$\boxed{B_{mn}=\frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy}$$

$$B_{mn}^{*}=\frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$