Chapter 4 Systems of ODEs. Phase Plane. Qualitative Methods

- higher-order ODEs $\longrightarrow$ system of 1^{st}-order ODEs
- solution of $\vec{y}' = A_{n\times n}\vec{y} + \vec{g}$, $A_{n\times n}$ — constant matrix
- qualitative method, phase plane, stability

Conversion of an n^{th}-order ODE to a system

$$y^{(n)}(t) = F\left(t, y, y', \cdots, y^{(n-1)}\right)$$

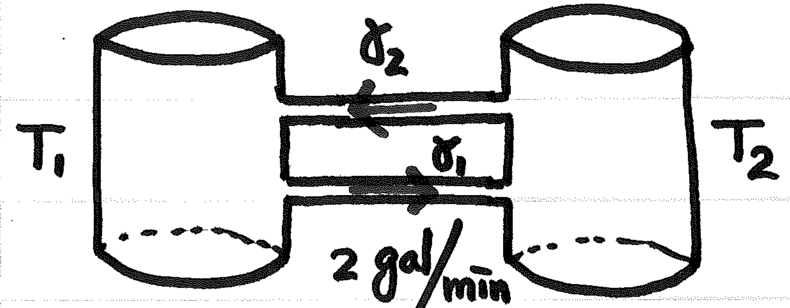
$$\begin{cases} y_1 = y \\ y_2 = y' \\ \vdots \\ y_n = y^{(n-1)} \end{cases} \Longrightarrow \vec{y}\,' = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}' = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ F(t, y_1, y_2, \cdots, y_n) \end{bmatrix}$$

Spring-Mass system $\quad my'' + cy' + ky = 0 \quad$ or $\quad y'' = -\frac{c}{m}y' - \frac{k}{m}y$

$$\begin{cases} y_1 = y \\ y_2 = y' = y_1' \end{cases} \qquad \begin{cases} y_1' = y_2 \\ y_2' = y_1'' = y'' = -\frac{c}{m}y' - \frac{k}{m}y = -\frac{c}{m}y_2 - \frac{k}{m}y_1 \end{cases} \qquad \vec{y}\,' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\vec{y}$$

§4.1 Systems of ODEs as Models in Engineering Applications

Ex. 1 (Mixing problems involving two tanks)



$Q_1 = 100$ gal $\quad Q_2 = 100$ gal

$y_i(t)$ —— the amount of fertilizer in T_i at time t

<u>Given</u> $\quad y_1(0) = 0, \; y_2(0) = 150$ lb

flow-rate $\quad \gamma_1 = 2$ gal/min, $\quad \gamma_2 = 2$ gal/min

<u>Question</u> $\quad t = ?$ s.t. $y_1(t) = \frac{1}{2} y_2(t)$

<u>Solution</u> $\quad$ <u>Math Model</u> $\quad y_i'(t) = $ inflow/min $-$ outflow/min

$$\begin{cases} y_1'(t) = \dfrac{y_2}{\underbrace{100}_{Q_2}} \gamma_2 - \dfrac{y_1}{\underbrace{100}_{Q_1}} \gamma_1 \\ y_2'(t) = \dfrac{y_1}{\underbrace{100}_{Q_1}} \gamma_1 - \dfrac{y_2}{\underbrace{100}_{Q_2}} \gamma_2 \end{cases}$$

$$\vec{y}' = A\vec{y} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix} \vec{y}$$

§4.2 Basic Theory of Systems of ODEs. Wronskian.

$$(1) \qquad \begin{cases} \vec{y}\,' = f(t, \vec{y}) \\ \vec{y}(t_0) = \vec{K} \end{cases}$$

$\vec{f}$, $\dfrac{\partial \vec{f}}{\partial t}$ are cont. in a region $R \ni (t_0, \vec{K}) \implies$ (1) has a unique solution $\vec{y}(t)$ on some interval $(t_0 - \alpha, t_0 + \alpha)$.

<u>Linear System</u>

$$(2) \qquad \begin{cases} \vec{y}\,' = A\vec{y} + \vec{g} \\ \vec{y}(t_0) = \vec{K} \end{cases}$$

$\vec{g} = 0$ homogeneous

$\vec{g} \neq 0$ non-homog.

$A = [a_{jk}(t)]$, $\vec{g}(t)$ are cont. on $(\alpha, \beta) \ni t_0$ $\implies$ (2) has a unique solution on (α, β).

homogeneous linear system

(3) $$\vec{y}\,' = \underset{n\times n}{A}\,\vec{y}$$

- $\vec{y}_1, \vec{y}_2$ – solutions of (3) $\Longrightarrow$ $c_1\vec{y}_1(t) + c_2\vec{y}_2(t)$ is solution of (3)
- $\{\vec{y}_1, \cdots, \vec{y}_n\}$ is a fundamental system of solutions of (3) $\Longleftrightarrow$ they are solutions of (3) and l. indep
- <u>general solution</u> $\vec{y}(t) = c_1\vec{y}_1(t) + \cdots + c_n\vec{y}_n(t)$

$$Y = [\vec{y}_1, \cdots, \vec{y}_n] \qquad W(\vec{y}_1, \cdots, \vec{y}_n) = \det Y$$

fundamental matrix   Wronskian

Homogeneous Systems with Const. Coeff.

$$\vec{y}\,' = A_{n\times n}\,\vec{y} \qquad A \text{ — const. matrix}$$

solution $\quad \vec{y} = e^{\lambda t}\vec{x} \qquad \vec{x}$ — const vector

$$\vec{y}\,' = \lambda e^{\lambda t}\vec{x}$$
$$A\vec{y} = e^{\lambda t}A\vec{x}$$
$$\Longrightarrow \quad A\vec{x} = \lambda\vec{x}$$

single eigenvalue

λ is a single eigenvalue of $A \Longrightarrow \vec{y} = e^{\lambda t}\vec{x}$

$\vec{x}$ is the corresp. eigenvector

$$\vec{y}' = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix} \vec{y}$$

$\lambda_1 = 0$

$\lambda_2 = -0.04$

$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

general solution $\vec{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-0.04t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

<u>IVP</u> $\vec{y}(0) = \begin{bmatrix} 0 \\ 150 \end{bmatrix} \Rightarrow \vec{y} = 75 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 75 e^{-0.04t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$