

## Homogeneous Systems with Const. Coeff.

$$\vec{y}' = A_{n \times n} \vec{y} \quad A - \text{const. matrix}$$

solution  $\vec{y} = e^{\lambda t} \vec{x} \quad \vec{x} - \text{const vector}$

$$\vec{y}' = \lambda e^{\lambda t} \vec{x} \implies A \vec{x} = \lambda \vec{x}$$
$$A \vec{y} = e^{\lambda t} A \vec{x}$$

## single eigenvalue

$\lambda$  is a single eigenvalue of  $A \implies \vec{y} = e^{\lambda t} \vec{x}$   
 $\vec{x}$  is the corresp. eigenvector

$$\vec{y}' = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix} \vec{y}$$

$$\lambda_1 = 0 \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\lambda_2 = -0.04$$

general solution  $\vec{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-0.04t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

IVP  $\vec{y}(0) = \begin{bmatrix} 0 \\ 150 \end{bmatrix} \implies \vec{y} = 75 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 75 e^{-0.04t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

## complex eigenvalues

$$\vec{y}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{y}, \quad 0 = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 + 1 \Rightarrow \lambda = 1 \pm i$$

$$\underline{\lambda_1 = 1+i} \quad -i x_1 + x_2 = 0 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{\lambda_2 = 1-i} \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{—real} \Rightarrow \vec{x}_2 = \overline{\vec{x}_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\vec{y} = c_1 e^{(1+i)t} \vec{x}_1 + c_2 e^{(1-i)t} \vec{x}_2$$

## real solution corresponding to complex eigenvalues

Euler formula  $e^{i\mu t} = \cos \mu t + i \sin \mu t$

eigenvalue  $\lambda \pm i\mu$

eigenvector  $\vec{a} \pm i\vec{b}$

solution  $e^{(\lambda + i\mu)t} (\vec{a} + i\vec{b}) = \vec{u}(t) + i\vec{v}(t)$

$$\vec{u}(t) = e^{\lambda t} (\vec{a} \cos \mu t - \vec{b} \sin \mu t)$$

$$\vec{v}(t) = e^{\lambda t} (\vec{a} \sin \mu t + \vec{b} \cos \mu t)$$

general solution  $\vec{y} = c_1 \vec{u}(t) + c_2 \vec{v}(t)$

## repeated eigenvalues

$$\vec{y}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \vec{y}, \quad 0 = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda-1)(\lambda-3) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

$$\underline{\lambda_1 = \lambda_2 = 2} \quad x_1 + x_2 = 0 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \vec{y}_1(t) = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{y}_2 = t e^{2t} \vec{x}_1 + e^{2t} \vec{\eta}$$

$$(A - 2I) \vec{\eta} = \vec{x}_1 \quad \begin{bmatrix} -1 & -1 & \vdots & 1 \\ 1 & 1 & \vdots & -1 \end{bmatrix} \Rightarrow \eta_1 + \eta_2 = -1 \Rightarrow \vec{\eta} = \begin{bmatrix} k \\ -1-k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \vec{y}_2 &= t e^{2t} \vec{x}_1 + e^{2t} \left( \begin{bmatrix} 0 \\ -1 \end{bmatrix} + k \vec{x}_1 \right) \\ &= t e^{2t} \vec{x}_1 + e^{2t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} + k e^{2t} \vec{x}_1 \end{aligned}$$

$$\vec{y}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \left( t e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$