Homogeneous Systems with Const. Coeff.

$$\vec{y}\,' = A_{n\times n}\vec{y} \qquad A - \text{const. matrix}$$

solution $\qquad \vec{y} = e^{\lambda t}\vec{x} \qquad \vec{x}$ - const vector

$$\begin{aligned} \vec{y}\,' &= \lambda e^{\lambda t}\vec{x} \\ A\vec{y} &= e^{\lambda t}A\vec{x} \end{aligned} \implies A\vec{x} = \lambda\vec{x}$$

single eigenvalue

λ is a single eigenvalue of $A \Longrightarrow \vec{y} = e^{\lambda t}\vec{x}$

$\vec{x}$ is the corresp. eigenvector

$$\vec{y}\,' = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix} \vec{y}$$

$\lambda_1 = 0$ $\quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = -0.04$

general solution $\quad \vec{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-0.04t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

<u>IVP</u> $\quad \vec{y}(0) = \begin{bmatrix} 0 \\ 150 \end{bmatrix} \Rightarrow \vec{y} = 75 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 75 e^{-0.04t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

complex eigenvalues

$$\boxed{A\vec{x}=\lambda\vec{x}}$$

$$\vec{y}'=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\vec{y},\quad 0=\begin{vmatrix}1-\lambda & 1\\ -1 & 1-\lambda\end{vmatrix}=(\lambda-1)^2+1 \Longrightarrow \underline{\lambda=1\pm i}$$

real

$\overline{\lambda}$ $\qquad$ $\overline{\vec{x}}$

$$\underline{\lambda_1=1+i}\qquad -ix_1+x_2=0 \Rightarrow \vec{x}_1=\begin{bmatrix}1\\ i\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}+i\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$\overline{A\vec{x}}=\overline{\lambda\vec{x}}$$

$$\underline{\lambda_2=1-i}\qquad A=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\text{—real} \Longrightarrow \vec{x}_2=\overline{\vec{x}_1}=\begin{bmatrix}1\\ 0\end{bmatrix}+i\begin{bmatrix}0\\ -1\end{bmatrix}$$

$$A\overline{\vec{x}}=\overline{A}\,\overline{\vec{x}}=\overline{\lambda}\,\overline{\vec{x}}$$

$$\vec{y}=c_1e^{(1+i)t}\vec{x}_1+c_2e^{(1-i)t}\vec{x}_2$$

$$e^t(\cos t+i\underline{\sin t})\left(\underline{\begin{bmatrix}1\\ 0\end{bmatrix}}+i\begin{bmatrix}0\\ 1\end{bmatrix}\right)$$

$$\vec{y}=c_1e^t\begin{bmatrix}\cos t-0\\ 0-\sin t\end{bmatrix}+c_2e^t\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

real solution corresponding to complex eigenvalues

Euler formula $e^{i\mu t} = \cos\mu t + i\sin\mu t$

eigenvalue $\lambda \pm i\mu$

eigenvector $\vec{a} \pm i\vec{b}$

solution $e^{(\lambda+i\mu)t}(\vec{a}+i\vec{b}) = \vec{u}(t) + i\vec{v}(t)$

$\vec{u}(t) = e^{\lambda t}(\vec{a}\cos\mu t - \vec{b}\sin\mu t)$

$\vec{v}(t) = e^{\lambda t}(\vec{a}\sin\mu t + \vec{b}\cos\mu t)$

general solution $\vec{y} = c_1\vec{u}(t) + c_2\vec{v}(t)$

$$e^{\lambda t}\cdot e^{i\mu t}(\vec{a}+i\vec{b})$$

$$= e^{\lambda t}\left[\vec{a}\cos\mu t - \vec{b}\sin\mu t\right]$$

$$+ e^{\lambda t}\left[\vec{b}\cos\mu t + \vec{a}\sin\mu t\right] i$$

repeated eigenvalues

$$\vec{y}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\vec{y}, \quad 0 = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (\lambda-1)(\lambda-3)+1 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

$ay''+by'+cy=0$ • $\lambda_1 \neq \lambda_2 \in R$

$a\lambda^2+b\lambda+c=0$ $\quad e^{\lambda_1 t}, e^{\lambda_2 t}$

• $\lambda = r \pm i\mu$

$e^{rt}\cos\mu t, \ e^{rt}\sin\mu t$

• $\lambda_1=\lambda_2=\lambda$

$e^{\lambda t}, \ \underline{te^{\lambda t}}$

$\lambda_1=\lambda_2=2 \qquad x_1 + x_2 = 0 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \vec{y}_1(t) = e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\vec{y}_2 = te^{2t}\vec{x}_1 + e^{2t}\vec{\eta}$$

$$\boxed{(A-2I)\vec{\eta} = \vec{x}_1} \qquad \left[\begin{array}{cc:c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right] \Rightarrow \eta_1 + \eta_2 = -1 \Rightarrow \vec{\eta} = \begin{bmatrix} k \\ -1-k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{y}_2' = \cancel{e^{2t}}(1+2t)\vec{x}_1 + 2\cancel{e^{2t}}\vec{\eta} = t\cancel{e^{2t}}A\vec{x}_1 + \cancel{e^{2t}}A\vec{\eta} = 2t\vec{x}_1 + A\vec{\eta}$$

$$\vec{x}_1 + 2\vec{\eta} = A\vec{\eta}$$

$$(A-2I)\vec{\eta} = \vec{x}_1$$

$$\vec{y}_2 = te^{2t}\vec{x}_1 + e^{2t}\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + k\vec{x}_1\right)$$

$$= te^{2t}\vec{x}_1 + e^{2t}\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \cancel{ke^{2t}\vec{x}_1}$$

$$\vec{y}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\left(te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$