# Graph Solutions in the Phase Plane

$\vec{y}'(t) = A_{2\times2}\,\vec{y}(t)$ solution $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ a curve in the $y_1y_2$-plane

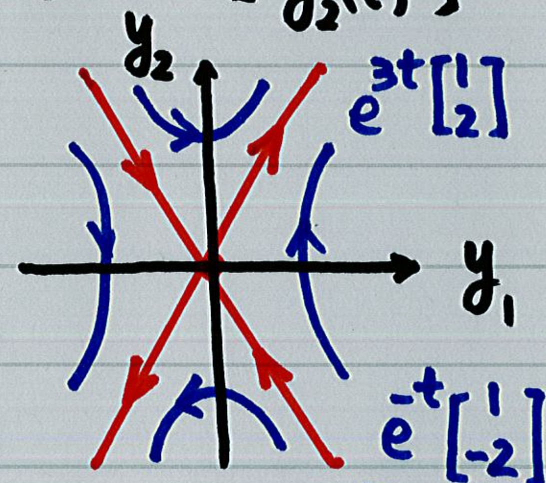
$t$ - parameter

$$\vec{y}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\vec{y}$$

$\lambda_1 = -1$

$\lambda_2 = 3$

$$\vec{y}(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

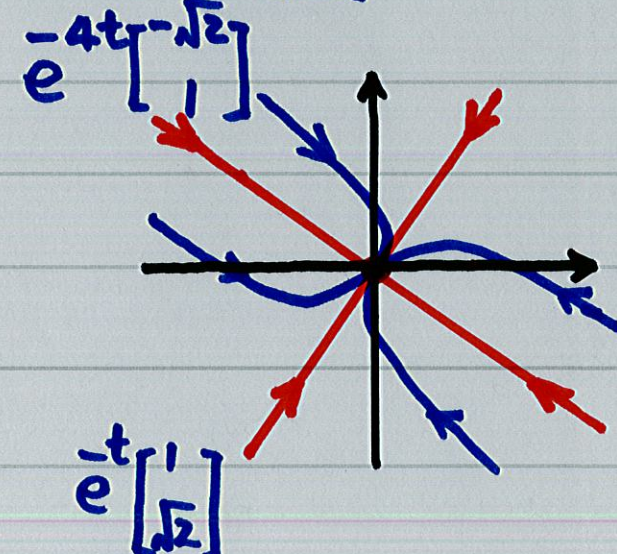



$$\vec{y}'(t) = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}\vec{y}$$

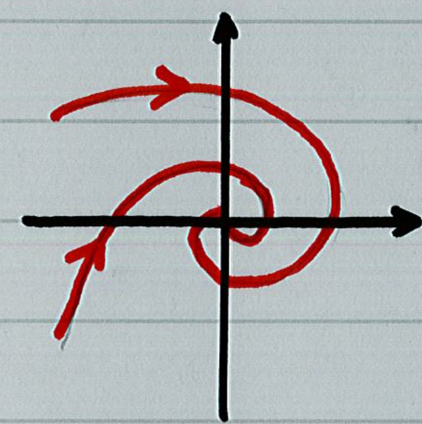
$\lambda_1 = -1$

$\lambda_2 = -4$

$$\vec{y}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} + c_2 e^{-4t}\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$



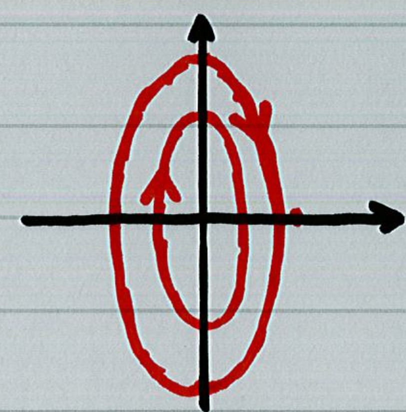
$$\vec{y}'(t) = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \vec{y}(t)$$



$$\lambda = -\frac{1}{2} \pm i, \quad \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{y} = c_1 e^{-\frac{1}{2}t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-\frac{1}{2}t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$$\vec{y}'(t) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \vec{y}$$



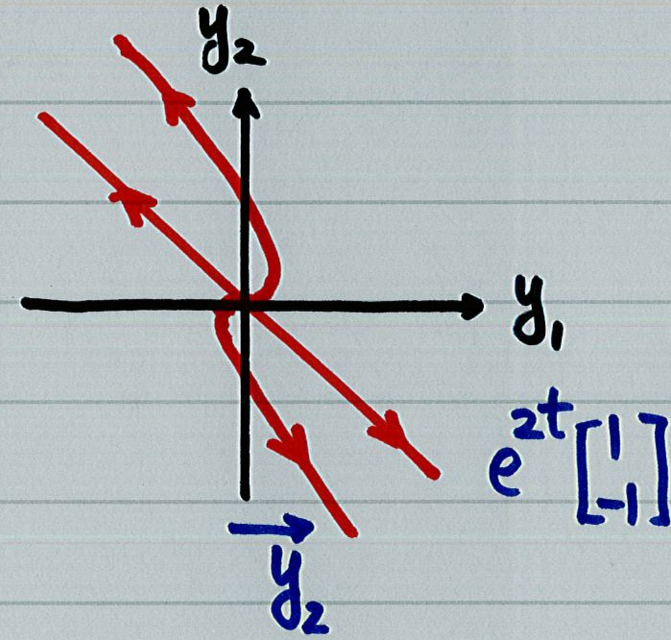
$$\lambda = \pm 2i \qquad \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\vec{y} = c_1 \begin{bmatrix} \cos 2t \\ -2\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix}$$

$$\vec{y}'(t) = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \vec{y}$$

$$\lambda_1 = \lambda_2 = 2 \qquad \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \vec{\eta} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\vec{y} = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \left\{ t e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$$



## §4.4 Criteria for Critical Pts. Stability.

$$\vec{y}\,'(t) = A_{2\times 2}\,\vec{y}(t)$$

eigenvalues: $\lambda_1, \lambda_2$

critical pts: $A\vec{y} = \vec{0}$ (if $|A| \neq 0$, then $\vec{y} = \vec{0}$ the only c.p.)

classification:

| | |
|---|---|
| Node | $\lambda_k \in R$ and $\lambda_1 \lambda_2 > 0$ |
| Saddle Pt | $\lambda_k \in R$ and $\lambda_1 \lambda_2 < 0$ |
| Center | $\lambda_k = \pm i\mu$ and $\mu \in R$ |
| Spiral Pt | $\lambda_k = r \pm i\mu$, $r \neq 0$ |

$$\begin{aligned} 0 &= |A - \lambda I| \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + |A| \\ &= \lambda^2 - p\lambda + q \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2) \\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \end{aligned}$$

$$p = \operatorname{tr} A = a_{11} + a_{22} = \lambda_1 + \lambda_2$$

$$q = |A| = \lambda_1 \lambda_2$$

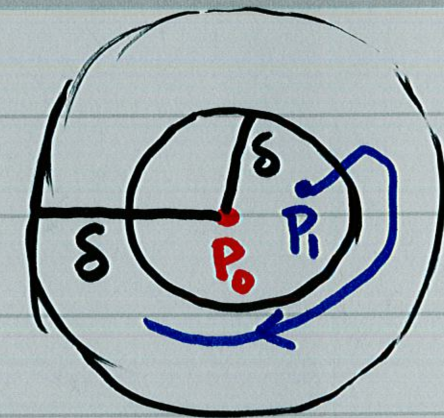
$$\lambda_{1,2} = \frac{p \pm \sqrt{\Delta}}{2}$$

$$\Delta = p^2 - 4q = (\lambda_1 - \lambda_2)^2$$

| | |
|---|---|
| Node | $\Delta \geqslant 0$ and $q > 0$ |
| Saddle Pt | $\Delta \geqslant 0$ and $q < 0$ |
| Center | $\Delta < 0$, $p = 0$, $q > 0$ |
| Spiral Pt | $\Delta < 0$ and $p \neq 0$ |

# stability

- A critical pt $P_0$ is stable



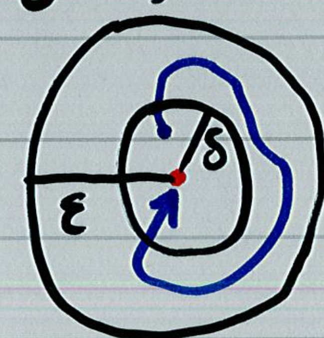

$$\Longleftrightarrow \forall \text{ disk } D_\varepsilon(P_0),\ \exists\ D_\delta(P_0) \subset D_\varepsilon(P_0)$$

s.t. every trajectory starting at $P_1 \in D_\delta(P_0)$ remains in $D_\varepsilon(P_0)$

- $P_0$ is stable and attractive

$\Longleftrightarrow$ every trajectory starting at $P_1 \in D_\delta(P_0)$

approaches to $P_0$ as $t \to \infty$



## #4 (p151)

$$\vec{y}\,' = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix} \vec{y}$$

## Ex. 2 (p151)  $my'' + cy' + ky = 0$

$$\vec{y}\,' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \vec{y}$$

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

$$p = -\frac{c}{m}, \quad q = \frac{k}{m}$$

$$\Delta = \frac{c - 4km}{m^2}$$

| | | $\Delta$ | $q$ | $p$ | |
|---|---|---|---|---|---|
| No damping | $c = 0$ | $-$ | $+$ | $0$ | center |
| underdamping | $c^2 < 4mk$ | $-$ | $+$ | $-$ | stable and attractive spiral pt |
| critical damping | $c^2 = 4mk$ | $0$ | $+$ | $-$ | stable and attractive node |
| overdamping | $c^2 > 4mk$ | $+$ | $+$ | $-$ | stable and attractive node |