§4.6 Nonhomogeneous Linear System of ODEs

$$\vec{y}\,'(t) = A\vec{y}(t) + \vec{g}$$

general solution $\quad \vec{y} = \vec{y}_h + \vec{y}_p$

$\vec{y}_h$ — general solution of homog. system

$\vec{y}_p$ — a particular solution of nonhomog. system

Method of Undetermined Coefficient

$$\vec{y}\,' = \begin{bmatrix}-3 & 1\\ 1 & -3\end{bmatrix}\vec{y} + \begin{bmatrix}-6\\ 2\end{bmatrix}e^{-2t} = A\vec{y}+\vec{g}$$

$$\vec{y}_h = c_1 e^{-2t}\begin{bmatrix}1\\ 1\end{bmatrix} + c_2 e^{-4t}\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\vec{y}_p = e^{-2t}\vec{v} + t e^{-2t}\vec{u}$$

$$\vec{y}_p\,' = -2\cancel{e^{-2t}}\vec{v} + \cancel{e^{-2t}}\vec{u} + (-2)\cancel{e^{-2t}}\,t\vec{u}$$

$$= \cancel{e^{-2t}}A\vec{v} + t\cancel{e^{-2t}}A\vec{u} + \begin{bmatrix}-6\\ 2\end{bmatrix}\cancel{e^{-2t}}$$

$$\left\{(A+2I)\vec{v} - \vec{u} + \begin{bmatrix}6\\ 2\end{bmatrix}\right\} + t\left\{(A+2I)\vec{u}\right\} = 0$$

$\{1, t\}$

$$\Rightarrow \begin{cases}(A+2I)\vec{u} = \vec{0}\\ (A+2I)\vec{v} = \vec{u} - \begin{bmatrix}-6\\ 2\end{bmatrix}\end{cases}$$

$$A+2I = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$$

$-u_1 + u_2 = 0$

$$\vec{u} = \begin{bmatrix}k\\ k\end{bmatrix} = k\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\left[\begin{array}{cc|c}-1 & 1 & k+6\\ 1 & -1 & k-2\end{array}\right] \quad \begin{bmatrix}-1 & 1 & k+6\\ 0 & 0 & 2k+4\end{bmatrix}$$

$2k+4=0$

$k=-2$

$-v_1+v_2=4$

$$\vec{u} = -2\begin{bmatrix}1\\ 1\end{bmatrix} \qquad \vec{v} = \begin{bmatrix}1\\ 5\end{bmatrix}$$

Method of Variation of Parameters

$$\vec{y}_h = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \equiv Y(t)\vec{c}$$

$$\left[e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix},\ e^{-4t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right]$$

$$\vec{y}_p = Y(t)\,\vec{u}(t)$$

$$Y'\vec{u} + Y\vec{u}' = AY\vec{u} + \vec{g}$$

$$\xrightarrow{Y' = AY} \quad \boxed{Y\vec{u}' = \vec{g}}$$

$$\vec{u}' = Y^{-1}\vec{g}$$

$$|Y| = 2e^{-6t}$$

$$Y^{-1}\vec{g} = \frac{1}{|Y|}\begin{bmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} e^{2t} & -e^{2t} \\ -e^{4t} & e^{4t} \end{bmatrix}\begin{bmatrix} 6e^{-2t} \\ 2e^{-2t} \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 6-2 \\ 6e^{2t}+2e^{2t} \end{bmatrix} = \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix} = \vec{u}'$$

$$u_1 = -2t + c_1$$

$$u_2 = -2e^{2t} + c_2$$

$$\vec{u} = -2\begin{bmatrix} t \\ e^{2t} \end{bmatrix} + \vec{c}$$