Chapter 6 Laplace Transform

$$\boxed{\text{IVP}} \xrightarrow{\mathcal{L}} \boxed{\text{algebraic eqs}} \longrightarrow \boxed{\text{solution of algebraic eq.}} \xrightarrow{\mathcal{L}^{-1}} \boxed{\text{solution of IVP}}$$

§6.1 Laplace Transform. Linearity. 1^{st} Shifting Thm (s-shifting).

$f(t)$ defined on $[0, \infty)$

$$F(s) = \underbrace{\mathcal{L}(f)}_{\text{Laplace transform}} = \int_0^\infty \underbrace{e^{-st}}_{\text{kernel}} f(t)\, dt$$

$f(t) = \mathcal{L}^{-1}(F)$ inverse Laplace transform

Table 6.1 $f(t)$ and $\mathcal{L}(f)$

$f(t)=\mathcal{L}^{-1}(F)$	1	t	t^2	t^n	$t^a\ (a>0)$	e^{at}
$F(s)=\mathcal{L}(f)$	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{2!}{s^3}$	$\frac{n!}{s^{n+1}}$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$\frac{1}{s-a}$

$f(t)=\mathcal{L}^{-1}(F)$	$\cos\omega t$	$\sin\omega t$	$\cosh at$	$\sinh at$
$F(s)=\mathcal{L}(f)$	$\frac{s}{s^2+\omega^2}$	$\frac{\omega}{s^2+\omega^2}$	$\frac{s}{s^2-a^2}$	$\frac{a}{s^2-a^2}$
$f(t)=\mathcal{L}^{-1}(F)$	$e^{at}\cos\omega t$	$e^{at}\sin\omega t$	$e^{at}f(t)$	$f(t-a)\,u(t-a)$
$F(s)=\mathcal{L}(f)$	$\frac{s-a}{(s-a)^2+\omega^2}$	$\frac{\omega}{(s-a)^2+\omega^2}$	$F(s-a)$	$e^{-as}F(s)$

$(fg)' = f'g + fg'$

$\boxed{fg = \int (fg)' = \int f'g + \int fg'}$

- $\underline{\mathcal{L}(1)} = \int_0^\infty e^{-st}\,dt = \dfrac{e^{-st}}{-s}\Big|_0^\infty = 0 - \dfrac{1}{-s} = \dfrac{1}{s}$

$\underline{s>0}$

$\boxed{a<s}$

- $\mathcal{L}(e^{at}) = \int_0^\infty e^{-st}\cdot e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \dfrac{e^{(a-s)t}}{a-s}\Big|_0^\infty = 0 - \dfrac{1}{a-s} = \dfrac{1}{s-a}$

$\cosh(at) = \dfrac{e^{at}+e^{-at}}{2}$

- $\mathcal{L}(\cosh at) = \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at}) = \frac{1}{2}\left[\dfrac{1}{s-a} + \dfrac{1}{s+a}\right] = \dfrac{s}{s^2-a^2}$

$\sinh(at) = \dfrac{e^{at}-e^{-at}}{2}$

$= \int_0^\infty e^{-st}[af+bg]\,dt = a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$

- $\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$

$\boxed{\int (f'g)\,dt = fg - \int fg'}$

- $\mathcal{L}(t^{n+1}) = \int_0^\infty e^{-st} t^{n+1}\,dt \;\dfrac{f' = e^{-st},\; g = t^{n+1}}{f = -\dfrac{e^{-st}}{s},\; g' = (n+1)t^n}\; - \dfrac{e^{-st}t^{n+1}}{s}\Big|_0^\infty + \dfrac{n+1}{s}\int_0^\infty e^{-st} t^n\,dt$

$$\lim_{t\to\infty} e^{-t}\cdot t^{n} = 0 = \lim_{t\to\infty}\frac{t^{n} \to \infty}{e^{t} \to \infty} = \lim_{t\to\infty}\frac{n t^{n-1}}{e^{t}}$$

$$= -\frac{1}{s}\left[e^{-s\infty}\,\infty^{n+1} - 0\right] + \frac{n+1}{s}\int_0^\infty e^{-st}t^{n}\,dt$$

$$= \frac{n+1}{s}\boxed{\int_0^\infty e^{-st}t^{n}\,dt} \qquad \underline{\mathcal{L}\{t^{n}\}}$$

$$= \mathcal{L}\{t^{n+1}\} = \frac{n+1}{s}\cdot\frac{n}{s}\,\mathcal{L}\{t^{n-1}\}\cdots = \frac{(n+1)\cdot n\cdots 1}{s^{n+2}}$$

$$\mathcal{L}\{\cos\omega t\} = \boxed{\int_0^\infty e^{-st}\cos\omega t\,dt} = \frac{s}{s^2+\omega^2}$$

$f' = e^{-st}$, $g = \cos\omega t$

$f = -\frac{e^{-st}}{s}$, $g' = -\omega\sin\omega t$

$$= -\frac{e^{-st}\cos\omega t}{s}\Big|_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st}\sin\omega t\,dt$$

$$= \frac{1}{s} - \frac{\omega}{s}\int_0^\infty e^{-st}\sin\omega t\,dt$$

$g = \sin\omega t$

$g' = \omega\cos\omega t$

$$= \frac{1}{s} - \frac{\omega}{s}\left[-\frac{e^{-st}\sin\omega t}{s}\Big|_0^\infty + \frac{\omega}{s}\overbrace{\int_0^\infty e^{-st}\cdot\cos\omega t\,dt}^{I}\right]$$

$$I = \frac{1}{s} - \frac{\omega^2}{s^2}I \qquad \left(1+\frac{\omega^2}{s^2}\right)I = \frac{1}{s}$$

$$I = \frac{s}{s^2+\omega^2}$$

S-shifting: $s \longrightarrow s-a$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

- $\mathcal{L}\{e^{at}\cos\omega t\} =$

  $\mathcal{L}\{e^{at}\sin\omega t\} =$

  $\mathcal{L}^{-1}\left\{\dfrac{3s-137}{s^2+2s+401}\right\} =$

- $f(t)$ is piecewise continuous and $|f(t)| \leq Me^{kt}\ \forall t \geq 0$ $\Longrightarrow$ $F(s) = \mathcal{L}(f)$ exists $\forall s > k$