§6.2 Transforms of Derivatives and Integrals. ODEs.

- $\boxed{\mathcal{L}(f') = s\mathcal{L}(f) - f(0)}$

  $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$

  $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$

Ex. $f(t) = t\sin\omega t$, $\qquad \mathcal{L}(f'') = s^2\underline{\mathcal{L}(f)} - s\underline{f(0)} - \underline{f'(0)}$

$$\boxed{\mathcal{L}(t\sin\omega t) = \frac{2\omega s}{(s^2+\omega^2)^2}} \qquad = s^2 \cdot \frac{2\omega s}{(s^2+\omega^2)^2}$$

$f(0) = 0$, $f'(t) = \sin\omega t + \omega t\cos\omega t$

$f'(0) = 0$

Ex. $f(t) = \cos\omega t$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0) = s^2 \frac{s}{s^2+\omega^2} - s$$

Laplace Transform of Integral of Function

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}(f) = \frac{1}{s}F(s)}$$

- $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \int_0^t \frac{1}{\omega}\sin\omega\tau\,d\tau$ $\qquad \mathcal{L}^{-1}\left\{F(s) = \frac{1}{s^2+\omega^2}\right\} = \frac{1}{\omega}\sin\omega t$

$$= -\frac{1}{\omega^2}\cos\omega\tau\Big|_0^t = -\frac{1}{\omega^2}\cos\omega t + \frac{1}{\omega^2}$$

IVP $\begin{cases} y'' + a y' + b y = r(t) \\ y(0) = K_0, \quad y'(0) = K_1 \end{cases}$

$$\mathcal{L}(r(t)) = \mathcal{L}(y'') + a\mathcal{L}(y') + b\mathcal{L}(y)$$

$$\underset{R(s)}{\parallel} \qquad = s^2 Y(s) - s y(0) - y'(0) \qquad \underset{Y(s)}{\parallel}$$

$$+ a\,[sY - y(0)] + bY$$

$$R(s) = [s^2 + as + b]\,Y(s) - [y(0)s + y'(0) + a y(0)]$$

$$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2}$$

transfer function

$$\boxed{Y(s) = [(s+a)y(0) + y'(0)]\,Q(s) + R(s)Q(s)} \Longrightarrow y(t) = \mathcal{L}^{-1}(Y)$$

$$Y(s) = \frac{R(s) + [(s+a)y(0) + y'(0)]}{s^2 + as + b}$$

- $\begin{cases} y'' - y = t \\ y(0) = 1, \quad y'(0) = 1 \end{cases}$

$$\boxed{s^2 Y - s y(0) - y'(0) - Y = \frac{1}{s^2}}$$

$$(s^2 - 1)\,Y = \frac{1}{s^2} + s + 1 \qquad \frac{Cs + D}{s^2}$$

$$s^2 - 1 = (s+1)(s-1)$$

$$\boxed{Y(s) = \frac{1}{s^2(s+1)(s-1)} + \frac{1}{s-1}}$$

$$= \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s} + \frac{D}{s^2} + \frac{1}{s-1}$$

$s=0$: $\quad 1=-D \Rightarrow D=-1$

$s=1$: $\quad 1=2B \Rightarrow B=\frac{1}{2}$

$s=-1$: $\quad 1=-2A \Rightarrow A=-\frac{1}{2}$

$s=2$:

$$1=4A+12B+6C+3D$$

$$=-2+6+6C-3$$

$$C=\frac{1}{6}[0]=0$$

$$\frac{1}{s^2(s+1)(s-1)}=\frac{1=As^2(s-1)+Bs^2(s+1)+Cs(s+1)(s-1)+D(s+1)(s-1)}{s^2(s+1)(s-1)}$$

$$1=\underbrace{[\quad]}_{=0}s^3+\underbrace{[\quad]}_{=0}s^2+\underbrace{[\quad]}_{=0}s+\underbrace{[\quad]}_{=1}$$

$$Y(s) = \frac{1}{s^2(s+1)(s-1)} + \frac{1}{s-1} \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$= \left[-\frac{1}{2} \cdot \frac{1}{s+1} + \frac{1}{2} \cdot \frac{1}{s-1} - \frac{1}{s^2} \right] + \frac{1}{s-1} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$= -\frac{1}{2}\frac{1}{s+1} + \frac{3}{2}\frac{1}{s-1} - \frac{1}{s^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\frac{1}{2}e^{-t} + \frac{3}{2}e^{t} - t$$

• (shifted data prob)

$$\begin{cases} y'' + y = 2t \\ y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \ y'(\tfrac{\pi}{4}) = 2-\sqrt{2} \end{cases} \Longrightarrow \begin{cases} \tilde{y}'' + \tilde{y}' = 2\tilde{t} + \tfrac{\pi}{2} \\ \tilde{y}(0) = \tfrac{\pi}{2}, \ \tilde{y}'(0) = 2-\sqrt{2} \end{cases}$$

$t_0 = \frac{\pi}{4}$

$t = \tilde{t} + \frac{1}{4}\pi$

$\tilde{y}(\tilde{t}) = y(t) \Rightarrow y'(t) = \dfrac{dy}{dt} = \dfrac{d\tilde{y}}{d\tilde{t}} \cdot \dfrac{d\tilde{t}}{dt} = \tilde{y}'(\tilde{t})$

$y''(t) = \tilde{y}''(\tilde{t})$

$$\frac{2}{s^2} + \frac{\pi}{2}\cdot\frac{1}{s} = \left[s^2\tilde{Y}(s) - s\,\tilde{y}(0) - \tilde{y}'(0)\right] + \left[s\tilde{Y}(s) - \tilde{y}(0)\right]$$

$$= (s^2+s)\tilde{Y}(s) - (s+1)\frac{\pi}{2} - (2-\sqrt{2})$$

$$Y(s) = \left[\frac{\pi}{2}\cdot\frac{1}{s} + \frac{2}{s^2} + \frac{\pi}{2}(s+1) + (2-\sqrt{2})\right]\frac{1}{s(s+1)}$$

$$= \frac{\pi}{2}\,\frac{1}{s^2(s+1)} + \frac{2}{s^3(s+1)} + \frac{\pi}{2}\cdot\frac{1}{s} + (2-\sqrt{2})\frac{1}{s(s+1)}$$

$\tilde{y}(\tilde{t}) = \mathcal{L}^{-1}\{Y(s)\}$