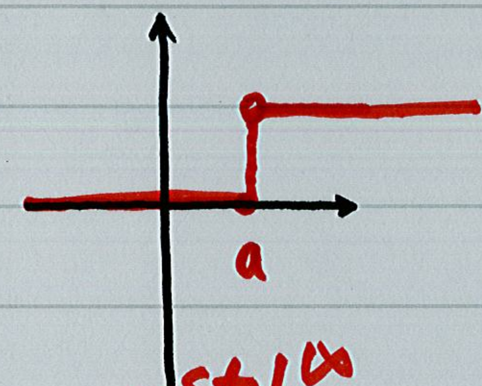
## §6.3 Unit Step Function. $2^{nd}$ Shifting Thrm ($t$-shifting).

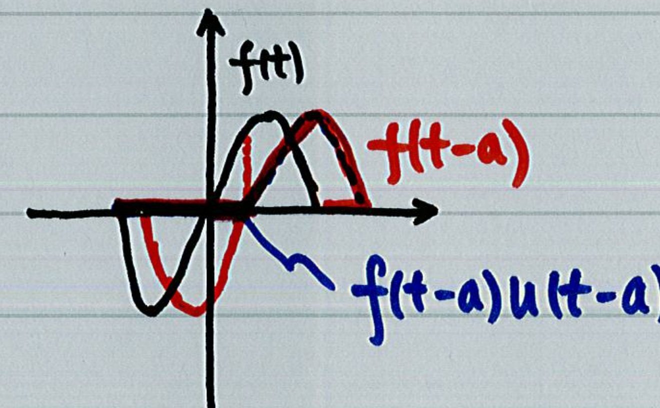
- unit step function

$$u(t-a)=\begin{cases}0 & t<a\\ 1 & t>a\end{cases}$$

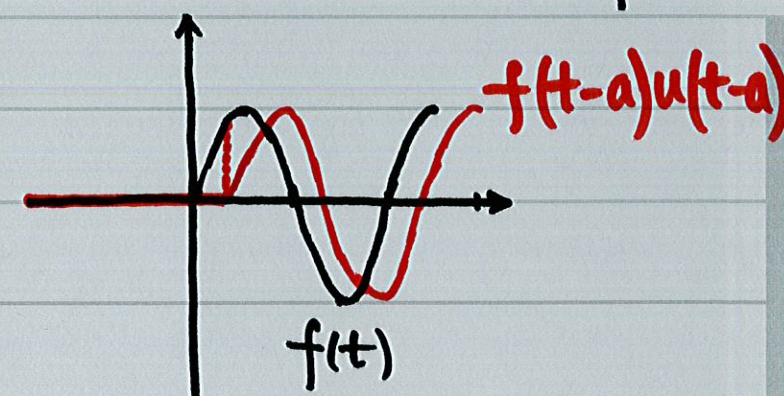



$$=\int_0^\infty e^{-st}u(t-a)\,dt=\int_0^a+\int_a^\infty=\int_a^\infty e^{-st}\,dt=\frac{e^{-st}}{-s}\Big|_a^\infty$$

- $$\boxed{\mathcal{L}\{u(t-a)\}=\frac{e^{-as}}{s}\quad (s>0)}$$




- $$f(t-a)u(t-a)=\begin{cases}0 & t<a\\ f(t-a) & t>a\end{cases}$$




shifting $f$ to the right by $a$

• $\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}\,\mathcal{L}(\overset{\text{" } F(s)}{f(t)})$

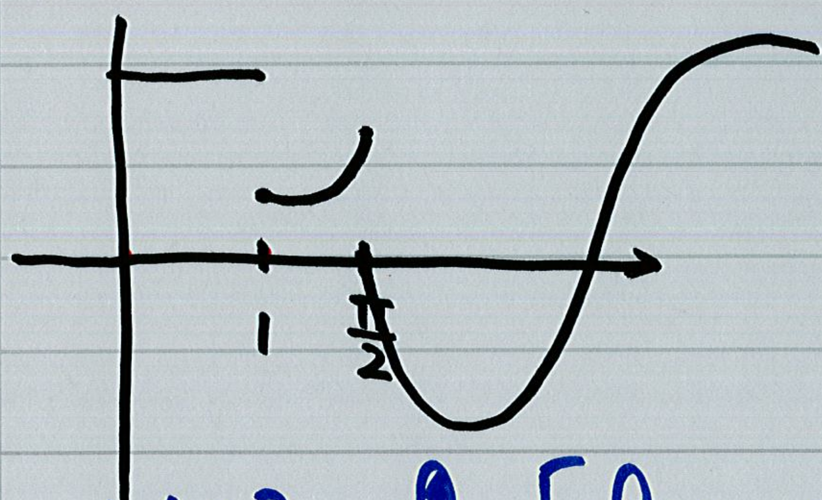
$\mathcal{L}^{-1}\{e^{-as}F(s)\}$
$= f(t-a)\,u(t-a)$

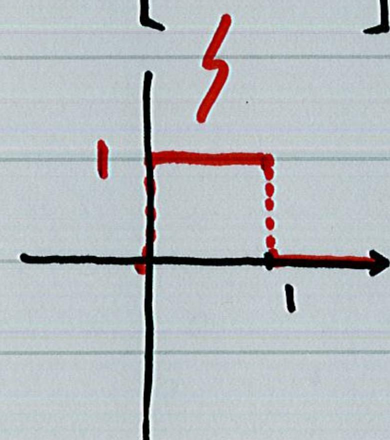
$\frac{\pi^2}{2\cdot 4} \approx 1.1$

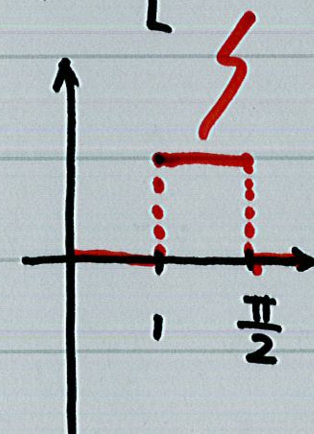
$\Rightarrow \mathcal{L}\{f(t)\,u(t-a)\} = \mathcal{L}\{f((t-a)+a)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$

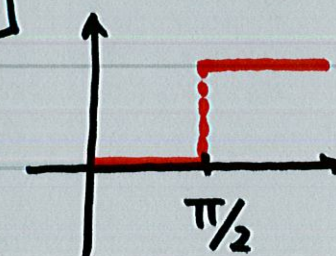
Ex. 1

$$f(t) = \begin{cases} 2 & 0<t<1 \\ \frac{1}{2}t^2 & 1<t<\frac{\pi}{2} \\ \cos t & t>\frac{\pi}{2} \end{cases} \overset{?}{=} 2\left[1-u(t-1)\right] + \frac{1}{2}t^2\left[u(t-\tfrac{\pi}{2}) - u(t-1)\right] + \cos t\cdot u(t-\tfrac{\pi}{2})$$




$\mathcal{L}\{f\} = \cancel{\mathcal{L}}\; 2\left[\mathcal{L}(1) - \mathcal{L}(u(t-1))\right] + \frac{1}{2}\left[\mathcal{L}(t^2 u(t-\tfrac{\pi}{2})) - \right.$

$$\mathcal{L}(f) = 2\left[\mathcal{L}(1) - \mathcal{L}\left(u(t-1)\right)\right] + \frac{1}{2}\left[\mathcal{L}\left(t^2 u(t-\tfrac{\pi}{2})\right) - \mathcal{L}\left(t^2 u(t-1)\right)\right] + \mathcal{L}\left(\underline{\cos t\, u(t-\tfrac{\pi}{2})}\right)$$

$$\left((t-\tfrac{\pi}{2}) + \tfrac{\pi}{2}\right)^2 = (t-\tfrac{\pi}{2})^2 + \pi\,(t-\tfrac{\pi}{2}) + \left(\tfrac{\pi}{2}\right)^2$$

$$\mathcal{L}\left(t^2 u(t-\tfrac{\pi}{2})\right) = \mathcal{L}\left((t-\tfrac{\pi}{2})^2 u(t-\tfrac{\pi}{2})\right) + \pi\,\mathcal{L}\left((t-\tfrac{\pi}{2})\, u(t-\tfrac{\pi}{2})\right) + \left(\tfrac{\pi}{2}\right)^2 \mathcal{L}\left(u(t-\tfrac{\pi}{2})\right)$$

$$= e^{-\frac{\pi}{2}s}\mathcal{L}(t^2) + \pi e^{-\frac{\pi}{2}s}\mathcal{L}(t) + \left(\tfrac{\pi}{2}\right)^2 e^{-\frac{\pi}{2}s} \cdot \frac{1}{s}$$

$$= \frac{2}{s^3}e^{-\frac{\pi}{2}s} + \frac{\pi}{s^2}e^{-\frac{\pi}{2}s} + \left(\frac{\pi}{2}\right)^2 \frac{1}{s} e^{-\frac{\pi}{2}s}$$

$$\mathcal{L}\left(t^2 u(t-1)\right) =$$

$$\mathcal{L}\left(\cos t\, u\left(t-\frac{\pi}{2}\right)\right) = -\mathcal{L}\left\{\sin\left(t-\frac{\pi}{2}\right) \underline{u\left(t-\frac{\pi}{2}\right)}\right\} = -e^{-\frac{\pi}{2}s}\mathcal{L}\{\sin t\}$$

$$\cos t = \cos\left(\left(t-\frac{\pi}{2}\right)+\frac{\pi}{2}\right) = \cos\left(t-\frac{\pi}{2}\right)\underbrace{\cancel{\cos\frac{\pi}{2}}}_{0} - \sin\left(t-\frac{\pi}{2}\right)\underbrace{\cancel{\sin\frac{\pi}{2}}}_{1} = -\sin\left(t-\frac{\pi}{2}\right)$$

$$= -e^{-\frac{\pi}{2}s}\frac{1}{s^2+1}$$

Ex. 2 $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+\pi^2}+\frac{e^{-2s}}{s^2+\pi^2}+\frac{e^{-3s}}{(s+2)^2}\right\}$ $\mathcal{L}^{-1}\{e^{-as}\underline{F(s)}\}=\underline{f}(t-a)\,u(t-a)$

$$=\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s^2+\pi^2}\right\}+\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s^2+\pi^2}\right\}+\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{(s+2)^2}\right\}$$

$$=\frac{1}{\pi}\sin\pi(t-1)\,u(t-1)+\frac{1}{\pi}\sin\pi(t-2)\,u(t-2)+e^{-2(t-3)}(t-3)\,u(t-3)$$

$$\mathcal{L}\{e^{at}f(t)\}=F(s-a)\Leftrightarrow\mathcal{L}^{-1}\{F(s-a)\}=e^{at}f(t)$$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\pi^2}\right\}=\frac{1}{\pi}\sin\pi t$ $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}=\underline{e^{-2t}\cdot t}$

$a=-2$