#25, p224

$$f(t-1) \qquad \underline{f=t}$$

$$\begin{cases} y''+y = \begin{cases} t & 0<t<1 \\ 0 & t>1 \end{cases} = t\left[1-u(t-1)\right] = t-(t-1)u(t-1) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -u(t-1) \\ y(0)=y'(0)=0 \end{cases}$$

$$s^2Y+Y = \frac{1}{s^2} - e^{-s}\frac{1}{s^2} - \frac{1}{s}e^{-s} = \frac{1}{s^2}\left(1-e^{-s}\right) - \frac{1}{s}e^{-s}$$

$$Y = \frac{1}{s^2+1}\left[\frac{1}{s^2}\left(1-e^{-s}\right) - \frac{1}{s}e^{-s}\right]$$

$$\frac{1}{(s^2+1)s^2} = \frac{1}{s^2} - \frac{1}{s^2+1}, \quad \frac{1}{(s^2+1)s} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\left(1-e^{-s}\right)\right\}$$

$$\mathcal{L}^{-1}\left\{\underbrace{\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)}\left(1-e^{-s}\right)\right\}$$

$$t \qquad \sin t$$

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)u(t-a)$$

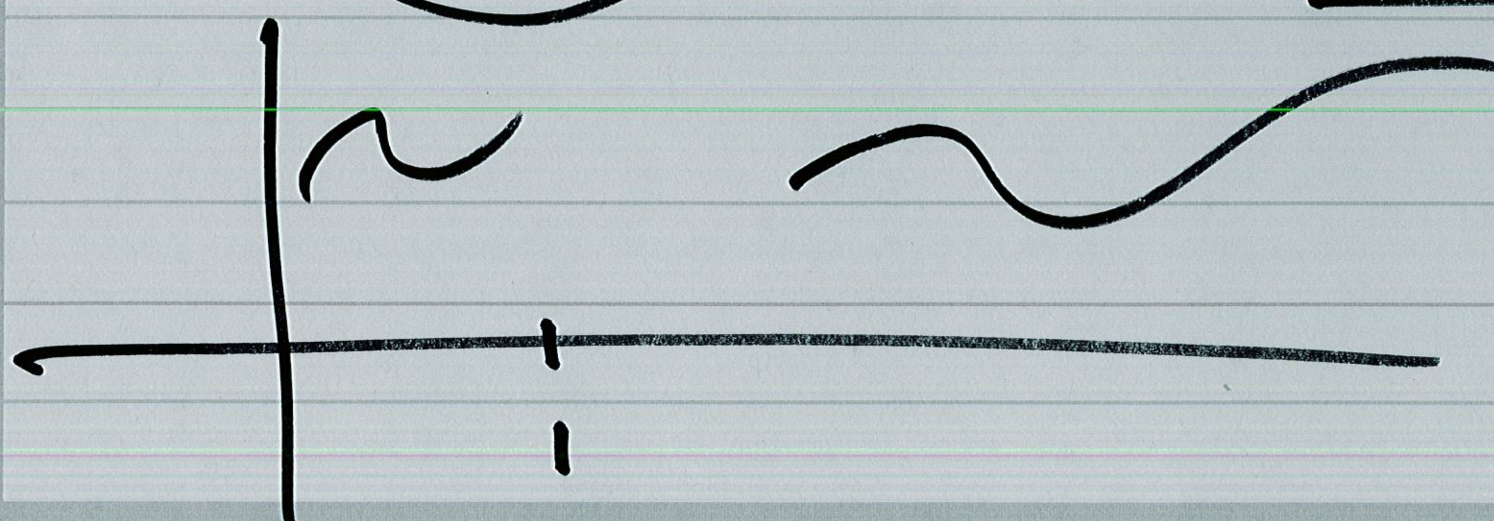
$$= \underline{(t-\sin t)-\left[(t-1)-\sin(t-1)\right]u(t-1)}$$

$$\mathcal{L}^{-1}\left\{\left(\frac{1}{s}-\frac{s}{s^2+1}\right)e^{-s}\right\} = \underline{\left[1-\cos(t-1)\right]u(t-1)}$$

$$1 \qquad \cos t$$

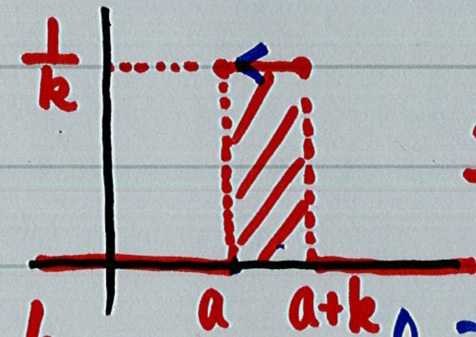
$$y(t) = \underline{(t-\sin t)} - \left[t-\sin(t-1)-\cos(t-1)\right]\underline{u(t-1)}$$

$$
\begin{aligned}
&= (t - \sin t)\left[1 - u(t-1)\right] + (t - \sin t)\, u(t-1) \\
&\qquad\qquad\qquad\qquad\qquad - \left[\qquad\qquad\right] u(t-1) \\
&= (t - \sin t)\left[1 - u(t-1)\right] + \left[\quad * \quad\right] u(t-1)
\end{aligned}
$$



§6.4 Short Impulses. Dirac's Delta Function. Partial Fractions.

$$f_k(t-a) = \begin{cases} \frac{1}{k} & a \le t \le a+k \\ 0 & \text{otherwise} \end{cases}$$



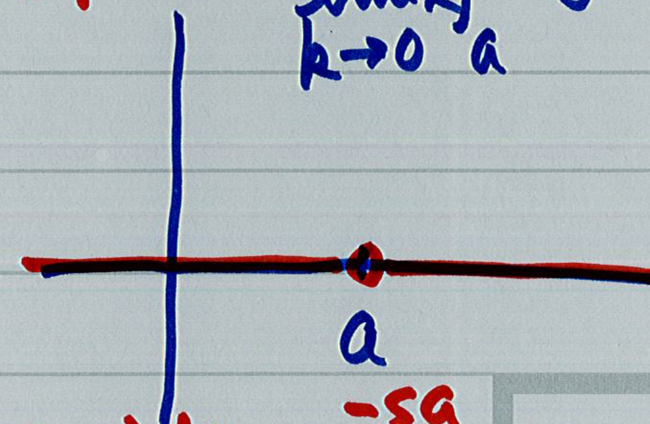
$1 = \text{area}$

$$\int_a^{a+k} \frac{1}{k}\, dt = \frac{1}{k}\cdot k = 1$$

$$I_k = \int_0^\infty f_k(t-a)\,dt = 1$$

$$\lim_{k\to 0} \frac{1}{k}\int_a^{a+k} g(t) = \frac{g(\xi)\cancel{k}}{} \rightarrow g(a)\cancel{k}$$

- **Dirac Delta Function**



$$\delta(t-a) = \lim_{k\to 0} f_k(t-a) = \begin{cases} \infty & t=a \\ 0 & \text{otherwise} \end{cases}$$

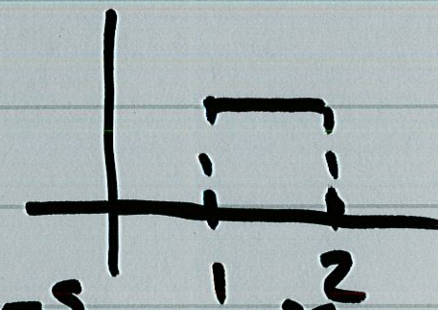
$$= \int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-sa}$$

$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a), \qquad \mathcal{L}\{\delta(t-a)\} = e^{-as}$$

$$= \lim_{k\to 0} \int_0^\infty g(t)\, f_k(t-a)\,dt = \lim_{k\to 0}\left[\int_a^{a+k} g(t)\,dt\right]\frac{1}{k}$$

Ex. 1

$$\begin{cases} y''+3y'+2y = r(t) = u(t-1)-u(t-2) \\ y(0)=0,\ y'(0)=0 \end{cases}$$



$$\left[s^2Y(s) - s\cancel{y(0)} - \cancel{y'(0)}\right] + 3\left[sY - \cancel{y(0)}\right] + 2Y = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$\boxed{(s^2+3s+2)}\,Y(s) = \frac{1}{s}\left(e^{-s} - e^{-2s}\right)$$

$$Y = \frac{1}{s(s+1)(s+2)}\left(e^{-s} - e^{-2s}\right)$$

$$y''+3y'+2y=0$$

$$y=e^{rt}$$

$$\boxed{r^2+3r+2}=0$$

$$=(r+1)(r+2)$$

$$y=c_1e^{-t}+c_2e^{-2t}$$

$$1 = A(s+1)(s+2)+Bs(s+2)+Cs(s+1)$$

$$\frac{1}{(\qquad)} = \frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

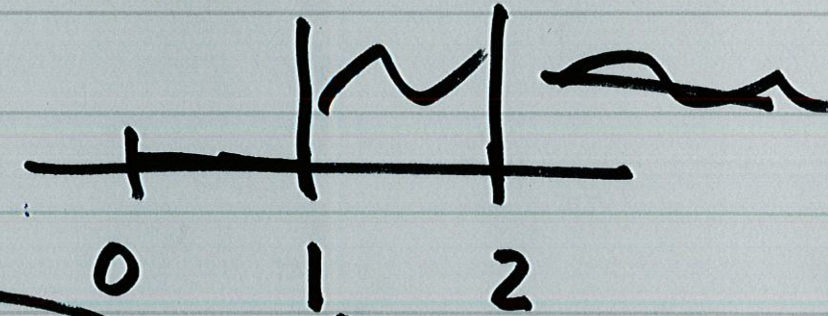
$\underline{s=0}$ $A=\frac{1}{2}$

$\underline{s=-1}$ $B=-1$

$\underline{s=-2}$ $C=\frac{1}{2}$

$$Y(s) = \left[\frac{1}{2}\,\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\,\frac{1}{s+2}\right]\left(e^{-s} - e^{-2s}\right)$$

$$\frac{1}{2} \qquad - e^{-t} \qquad + \frac{1}{2}e^{-2t}$$



$$y(t) = \mathcal{L}^{-1}\{Y\}$$

$$= \left[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right]u(t-1)$$

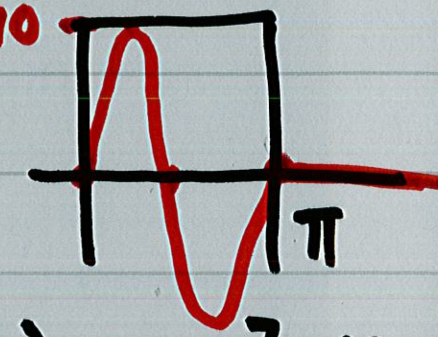
$$- \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]u(t-2)$$

$$\left[u(t-1) - u(t-2)\right] + u(t-2)$$

Ex. 2

$$\begin{cases} y'' + 3y' + 2y = \delta(t-1) \\ y(0) = 0,\ y'(0) = 0 \end{cases}$$

Ex. 4 $\quad \begin{cases} y''+2y'+2y = r(t) \\ y(0)=1,\ y'(0)=-5 \end{cases}$ with $r(t)=\begin{cases} 10\sin 2t & 0<t<\pi \\ 0 & t>\pi \end{cases}$



$$= 10\sin 2t\,\left[1-u(t-\pi)\right]$$

$$= 10\sin 2t - 10\sin\left[2(t-\pi)+2\pi\right]u(t-\pi)$$

$$= 10\sin 2t - 10\sin 2(t-\pi)\,u(t-\pi)$$

$f(t-\pi)\,u(t-\pi)$

$$(s^2Y - s + 5) + 2(sY-1) + 2Y$$

$$= (s^2+2s+2)Y - (s-3)$$

$$Y = \frac{1}{(s+1)^2+1}\left[(s-3) + 10\cdot\frac{2}{s^2+2^2} - 10e^{-\pi s}\frac{2}{s^2+2^2}\right]$$

$$= \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\cdot 4$$

$$+10\,\boxed{\frac{1}{(s+1)^2+1}\cdot\frac{2}{s^2+2^2}}\left(1-e^{-\pi s}\right)$$