§ 6.5 Convolution. Integral Equation.

$$h(t) = (f*g)(t) \equiv \int_0^t f(\tau)\, g(t-\tau)\, d\tau \overset{?}{=} \int_0^t g(\tau)\, f(t-\tau)\, d\tau = g*f$$

$\tau = t - \xi \quad t - \tau = \xi : t \to 0 \quad -\int_t^0 f(t-\xi)\, g(\xi)\, d\xi$

$d\tau = -d\xi$

- **Properties**

$f*g = g*f$ commutative

$f*(g_1+g_2) = f*g_1 + f*g_2$ distributive

$(f*g)*h = f*(g*h)$ associative

$f*0 = 0*f = 0$

$\int_0^t f(\tau)\, d\tau = \boxed{f*1 \neq f}$

$$\mathcal{L}\{f*g\} = F(s)G(s) \iff \mathcal{L}^{-1}\{F\cdot G\} = (f*g)(t)$$

$$\mathcal{L}^{-1}\{F\}\mathcal{L}^{-1}\{G\} \neq fg$$

<u>Ex. 1</u> $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)s}\right\} = \int_0^t e^{a\tau}\cdot 1\,d\tau = \frac{1}{a}e^{a\tau}\Big|_0^t$

$e^{at} * 1$

$$= \frac{1}{a}\left[e^{at}-1\right]$$

$$\frac{1}{(s-a)s} = \left[\frac{1}{s-a} - \frac{1}{s}\right]\frac{1}{a}$$

$$= \frac{1}{a}\left[e^{at}-1\right]$$

<u>Ex. 2</u> $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+\omega^2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\cdot\frac{1}{s^2+\omega^2}\right\}$

$$= \frac{1}{\omega^2}\int_0^t \sin\omega\tau \cdot \sin\omega(t-\tau)\,d\tau$$

$\frac{1}{\omega}\sin\omega t * \frac{1}{\omega}\sin\omega t$

$$= \frac{1}{2\omega^2}\left[-t\cos\omega t + \frac{\sin\omega t}{\omega}\right]$$

$$\sin x \sin y = \frac{1}{2}\left[-\cos(x+y) + \cos(x-y)\right]$$

Application to Nonhomogeneous Linear ODEs

$R(s) = \mathcal{L}(r)$

$$\begin{cases} y'' + ay' + by = r(t) \\ y(0) = 0, \quad y'(0) = 0 \end{cases}$$

a, b — constant

$$y(t) = \int_0^t q(t-\tau)\, r(\tau)\, d\tau$$

$$[s^2 Y - \cancel{s y(0)} - \cancel{y'(0)}] + a[sY - \cancel{y(0)}] + bY = R(s)$$

$$q(t) = \mathcal{L}^{-1}\{Q(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + as + b}\right\}$$

$$\boxed{(s^2 + as + b)}\, Y = R(s)$$

$$Q(s) = \frac{1}{s^2 + as + b} \qquad q = \mathcal{L}^{-1}\{Q\}$$

$$Y(s) = R(s)\, Q(s)$$

$$y(t) = \mathcal{L}^{-1}\{R(s) Q(s)\} = r * q = \boxed{\int_0^t r(\tau)\, q(t-\tau)\, d\tau}$$

Ex.4 $\begin{cases} y'' + \omega_0^2 y = K \sin \omega_0 t \\ y(0) = y'(0) = 0 \end{cases}$ with $\omega_0^2 = \dfrac{k}{m}$

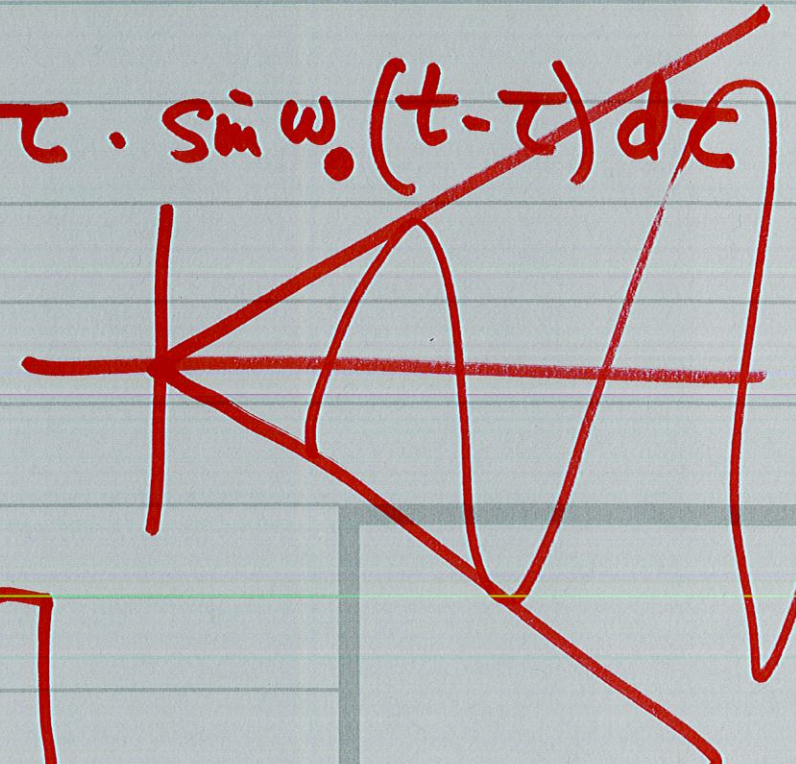
$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega_0^2}\right\} = \frac{1}{\omega_0}\sin\omega_0 t$$

$$y(t) = \cancel{\mathcal{L}^{-1}}\left\{\cancel{\overset{Q(s)}{K\sin\omega_0 t}\ \overset{R(s)}{g(t)}}\right\}$$

$$= \underline{K\sin\omega_0 t} * g(t) = \frac{K}{\omega_0}\int_0^{\tau} \sin\omega_0\tau \cdot \sin\omega_0(t-\tau)\,d\tau$$

$$= \frac{K}{\omega_0}\,\frac{1}{2}\left[-t\cos\omega_0 t + \frac{\sin\omega_0 t}{\omega_0}\right]$$

$$= \frac{K}{2\omega_0}\left[-t\,\omega_0\cos\omega_0 t + \sin\omega_0 t\right]$$



Ex. 5 $\begin{cases} y''+3y'+2y = r \\ y(0)=y'(0)=0 \end{cases}$

$$r(t) = \begin{cases} 1 & 0<t<2 \\ 0 & t>2 \end{cases} \qquad r(t) = \begin{cases} 0 & 0<t<1 \\ 1 & 1<t<2 \\ 0 & t>2 \end{cases}$$

$$g(t) = \mathcal{L}^{-1}\left\{ \frac{1}{\underbrace{s^2+3s+2}_{(s+1)(s+2)}} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{(s+1)(s+2)} \right\}$$

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$= e^{-t} - e^{-2t}$$

$$e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)}$$

$$\underline{y(t)} = \boxed{r(t) * g(t)} = g * r = \int_0^t \left(e^{-\tau} - e^{-2\tau}\right) \underline{r(t-\tau)}\, d\tau$$

$$= \int_0^t \underline{\underline{r(\tau)}} \left[e^{-(t-\tau)} - e^{-2(t-\tau)} \right] d\tau$$

$$= \begin{cases} \displaystyle\int_0^t \left[e^{-(t-\tau)} - e^{-2(t-\tau)} \right] d\tau & 0<t<2 \\ \displaystyle\int_0^2 \left[e^{-(t-\tau)} - e^{-2(t-\tau)} \right] d\tau + \int_2^t \cancel{0 \cdot [\ \ ]} & t>2 \end{cases}$$

$$y(t) = \begin{cases} \left[e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)} \right]_{\tau=0}^{\tau=t} & 0 < t < 2 \\ \left[e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)} \right]_{\tau=0}^{\tau=2} & t > 2 \end{cases}$$

$$= \begin{cases} \boxed{\frac{1}{2} - \left[e^{-t} - \frac{1}{2} e^{-2t} \right] \quad 0 < t < 2} \\ \left[e^{-(t-2)} - \frac{1}{2} e^{-2(t-2)} \right] - \left[e^{-t} - \frac{1}{2} e^{-2t} \right] \\ \cancel{\frac{1}{2} - \left[e^{-(t-2)} - \frac{1}{2} e^{-2(t-2)} \right]} & t > 2 \end{cases}$$

- Integral Equation of 2^{nd} kind

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)\,d\tau = t$$

$$\mathcal{L}\{y\} - \mathcal{L}\{y(t) * \sin(t)\} = \mathcal{L}(t)$$

$$Y - Y\frac{1}{s^2+1} = \frac{1}{s^2}$$

$$Y\left[\frac{s^2}{s^2+1}\right] = \frac{1}{s^2}$$

$$Y(s) = \frac{s^2+1}{s^2}\cdot\frac{1}{s^2} = \frac{s^2}{s^4} + \frac{1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= t + \frac{t^3}{3!}$$

$$= t + \frac{1}{6}t^3$$