#14 p231

$f(t) = f(t+p) \quad \forall t \qquad 1 + e^{-sp} + e^{-s2p} + \cdots$

$$\mathcal{L}(f) \overset{?}{=} \frac{1}{1-e^{-ps}} \boxed{\boxed{\int_0^p} e^{-st} f(t)\,dt}$$

$$\int_0^\infty = \int_0^p + \int_p^{2p} + \cdots$$

$$\parallel$$

$$\int_0^\infty e^{-st} f(t)\,dt$$

$$\boxed{e^{-st}} \cdot e^{-sp}$$

e^{-2ps}

$0 \quad p \quad 2p$

$$\int_p^{2p} e^{-st} f(t)\,dt \longrightarrow \int_0^p e^{-s(\tau+p)} \underbrace{f(\tau+p)}_{\boxed{f(t)}} d\tau = e^{-sp} \boxed{\int_0^p e^{-st} f(t)\,dt}$$

$\tau = t - p$: $0 \to p$

$d\tau = dt$

- **Integral Equation of 2^{nd} kind**

$$y(t) - \int_0^t y(\tau)\sin(t-\tau)\,d\tau = t$$

$$\mathcal{L}\{y\} - \mathcal{L}\{y(t) * \sin(t)\} = \mathcal{L}(t)$$

$$Y - Y\frac{1}{s^2+1} = \frac{1}{s^2}$$

$$Y\left[\frac{s^2}{s^2+1}\right] = \frac{1}{s^2}$$

$$Y(s) = \frac{s^2+1}{s^2}\cdot\frac{1}{s^2} = \frac{s^2}{s^4} + \frac{1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= t + \frac{t^3}{3!}$$

$$= t + \frac{1}{6}t^3$$

§6.6 Differentiation and Integration of Transforms. ODEs with Variable Coefficients.

- **Differentiation**

$$F'(s) = \int_0^\infty \frac{d}{ds}\left(e^{-st} f(t)\right) dt = -\int_0^\infty t e^{-st} f(t)\, dt$$

$$\mathcal{L}(f) = F(s) = \int_0^\infty e^{-st} f(t)\, dt \Longrightarrow F'(s) = -\int_0^\infty e^{-st} t f(t)\, dt = -\mathcal{L}(t f(t))$$

$$\mathcal{L}^{-1}\{F'(s)\} = -t f(t) \qquad F'(s) = -\mathcal{L}(tf)$$

$$\mathcal{L}\{t \sin\beta t\} = -\left(\frac{\beta}{s^2+\beta^2}\right)' = -\beta\left((s^2+\beta^2)^{-1}\right)' = \beta(s^2+\beta^2)^{-2}\cdot 2s = \frac{2s\beta}{(s^2+\beta^2)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+\beta^2)^2}\right\} = \frac{1}{2\beta} t \sin\beta t$$

$$\mathcal{L}\{t \cos\beta t\} = -\left(\frac{s}{s^2+\beta^2}\right)' = -\frac{(s^2+\beta^2) - s(2s)}{(s^2+\beta^2)^2}$$

$$= \frac{\beta^2 - s^2}{(s^2+\beta^2)^2}$$

- <u>Integration</u>

$$\int_s^{\infty}\int_0^{\infty} e^{-\hat{s}t} f(t)\,dt$$

$$\|$$

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt \Longrightarrow \cancel{G(s)}\int_s^{\infty} F(\hat{s})\,d\hat{s} = \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_0^{\infty} e^{-st}\frac{f(t)}{t}\,dt$$

$$= F(s)$$

$0 < t < \infty \qquad < \hat{s} <$

$s < \hat{s} < \infty$

$$\mathcal{L}^{-1}\left\{\ln\frac{s^2+w^2}{s^2}\right\} =$$

$$\mathcal{L}^{-1}\{F'(s)\} = -t\,f(t) \qquad \underline{F'(s)} = -\mathcal{L}\{t\,f(t)\}$$

$$\cancel{f(s) \neq G(s)}$$

$$F'(s) = \left(\ln(s^2+w^2) - \ln s^2\right)' = \frac{2s}{s^2+w^2} - \frac{2s}{s^2}$$

$$= 2\cdot\frac{s}{s^2+w^2} - 2\cdot\frac{1}{s}$$

$$\mathcal{L}^{-1}\{F'(s)\} = 2\cos wt - 2 = -t\,f(t) = -t\,\boxed{\mathcal{L}^{-1}\{F(s)\}}$$

$$\boxed{\mathcal{L}^{-1}\{F\} = -\frac{\mathcal{L}^{-1}\{F'(s)\}}{t}} = -\frac{2\cos\omega t - 2}{t}$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\hat{s})\,d\hat{s} \iff \frac{f(t)}{t} = \mathcal{L}^{-1}\left\{\int_s^\infty F(\hat{s})\,d\hat{s}\right\}$$

$$\mathcal{L}^{-1}(F(s)) = f(t) = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(\hat{s})\,d\hat{s}\right\}$$

$$\boxed{\mathcal{L}^{-1}(F(s)) = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(\hat{s})\,d\hat{s}\right\}}$$

$$F(s) = \ln\frac{s^2+\omega^2}{s^2}$$

$$\int_s^\infty \ln\left(\frac{\hat{s}^2+\omega^2}{\hat{s}^2}\right) d\hat{s}$$

$$\mathcal{L}\{ty'\} = -Y(s) - sY'(s), \quad \mathcal{L}\{t^2 y''\} = -2sY(s) - s^2 Y'(s) + y(0)$$

Ex. $ty'' + (1-t)y' + ny = 0$